

Operator calculus of differential chains and differential forms

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Abstract

We describe a topological predual \mathcal{B}' to the Fréchet space of differential forms \mathcal{B} defined in an open subset $U \subset \mathbb{R}^n$. This proper subspace of currents \mathcal{B}' has useful properties: Subspaces of finitely supported Dirac chains and polyhedral chains are dense, offering a unification of discrete and continuum viewpoints. Operators can be defined constructively and geometrically on Dirac chains of arbitrary dimension and dipole order. The operators are continuous, and are thus defined on limits of Dirac chains, including polyhedral chains, submanifolds, stratified sets, and fractals. The operator algebra contains operators predual to exterior derivative, Hodge star, Lie derivative, wedge and interior product on differential forms, yielding simplifications and extensions of the classical integral theorems of calculus including theorems of Stokes, Gauss-Green, and Kelvin-Stokes to arbitrary dimension and codimension. The limit chains, called “differential chains” may be highly irregular, and the differential forms may be discontinuous across the boundary of U . We announce new fundamental theorems for nonsmooth domains and their boundaries evolving in a flow. We close with broad generalizations of the Leibniz integral rule and Reynolds’ transport theorem.

1 Introduction

Infinite dimensional linear spaces can offer insights and simplifications of analytical problems, since continuous linear operators acting on the space can appear as nonlinear operations on an associated object, such as a finite dimensional manifold. A simple example is pushforward of the time- t map of the flow of a Lipschitz vector field as seen, for example, in the linear space of Whitney’s sharp chains, or in the space of de Rham currents if the vector field is smooth with compact support. There are numerous ways for creating such spaces, and each has different properties. Although Whitney’s Banach spaces of sharp and flat chains each have useful operators, no single Banach space carries an operator algebra sufficient for the most interesting applications. Federer and Fleming’s normal and integral currents afford a stronger calculus in many ways, but their use of the flat norm carries the typical flat problems such as lack of continuity of the Hodge star operator. The hallmark application of their seminal paper is the celebrated Plateau’s Problem of the calculus of variations, but their solutions do not permit Möbius strips or any surfaces with triple junctions. The author has treated Plateau’s Problem (in its more general setting of soap films arising in nature, permitting non-orientability and branchings) as a “test problem” for the depth and flexibility of

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any new calculus of variations. Long was the search for a linear space which possessed the right balance of size (the smaller the better so as to avoid pathologies,) and operator algebra richness (containing the operators relevant to calculus such as boundary). This paper covers some of the main properties of the space which ultimately succeeded, and the aforementioned application is contained in a sequel [Har04b]. In this paper, we set up the theory for chains defined in an open set and extend Stokes' Theorem 6.7.2 to this setting. We prove higher order divergence theorems for net flux of k -vector fields across nonsmooth boundaries in Corollary 3.6.9. We state and prove a new fundamental theorem of calculus for evolving chains and their boundaries Theorem 10.2.1. Finally, we present a generalization of the classical "differentiation of the integral" for nonsmooth evolving differential chains and forms Theorem 10.3.3 from which follows a broad generalization of the powerful Reynolds' transport theorem 10.3.6.

1.1 Dirac chains and the Mackey topology

Mackey set the stage for the study of topological predual spaces with his seminal papers [Mac43, Mac44] when he and Whitney were both faculty at Harvard. Most relevant to us is the following result: Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a dual pair¹ separated in Y . According to the Mackey-Arens Theorem [Are47], there exists a finest locally convex topology τ on X , called the *Mackey topology*, such that the continuous dual X' is the image of Y under the injection $i_Y : Y \rightarrow X^*$. The weak topology on X is the coarsest such topology. In particular, the Mackey topology does not make linear functions continuous which were discontinuous in the weak topology.

The Mackey topology is canonical and rather beautiful in its abstract conception, but it is often useful to find explicit formulations of it. For example, the space of polyhedral k -chains can be paired with any space of continuous differential k -forms via integration, yielding a dual pair separated in its second component. The simplest example is given by the Banach space of continuous and bounded differential k -forms on \mathbb{R}^n . The Mackey topology on polyhedral k -chains turns out to be the classical mass norm. When paired with Lipschitz forms, the Mackey topology on polyhedral chains is Whitney's sharp norm, originally called the "tight norm" topology in his 1950 ICM lecture [Whi50]. However, the boundary operator is not continuous in the sharp norm topology, which is something Whitney needed for his study of sphere bundles². He then defined the flat norm which does have a continuous boundary operator, and his student Wolfe [Wol48] identified the topological dual to flat chains. The flat topology is limited because the Hodge star operator is not closed.

Given these problems with the above established spaces, it became clear to the author that a new space was needed. Our main requirements for such a space are as follows: It should have the Mackey topology given a space of differential k -forms, and operators predual to classical operators on forms such as exterior derivative, Lie derivative, Hodge star, pullback, and interior product should be well-defined and continuous. Furthermore, the space should be as small as possible, and possess good topological properties. In particular it should be Hausdorff, separable, and sequentially complete, if not complete. The main goal of this paper is to define such a space, exhibit some of its properties, and provide a few applications.

Let U be an open subset of \mathbb{R}^n . (An extension of the theory to open subsets of a Riemannian manifold M is in progress. See §11 for the main ideas.) For $0 \leq k \leq n$, define a "Dirac k -chain"³ in U to be a finitely supported

¹A *dual pair* $(X, Y, \langle \cdot, \cdot \rangle)$ is a pair of vector spaces over \mathbb{R} together with a bilinear form $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{R}$. The dual pair is separated in Y (resp. X) if the map $i_Y : Y \rightarrow X^*$ induced by $\langle \cdot, \cdot \rangle$ (resp. $i_X : X \rightarrow Y^*$) is injective.

²According to his own account in [Whi88], Whitney wanted to solve a foundational question about sphere bundles and this is why he developed Geometric Integration Theory. Steenrod [Ste51] solved the problem first, though, and Whitney stopped working on GIT (see [Whi88]). However, Whitney's student Eells was interested in other applications of GIT in analysis, and for a time he thought he had a workaround (see [Whi57] and [Eel55]), but the lack of a continuous boundary operator halted progress.

³Previously called a "pointed k -chain". We prefer not to use the term "Dirac current" since it has various definitions in the literature, and the whole point of this work is to define calculus starting with chains, setting aside the larger space of currents.

section of the k -th exterior power of the tangent bundle of U .

We can write a Dirac k -chain as a formal sum $A = \sum_{i=1}^s (p_i; \alpha_i)$ where $p_i \in U$ and $\alpha_i \in \Lambda_k(T_{p_i}(U))$. We call $(p; \alpha)$ a *simple k -element* if α is a simple k -vector, otherwise $(p; \alpha)$ is a *k -element*. The *support* of a Dirac chain $\sum_{i=1}^s (p_i; \alpha_i)$ is defined as $\text{supp}(\sum_{i=1}^s (p_i; \alpha_i)) := \cup_{i=1}^s p_i$. We say that a Dirac chain A is *supported in* $X \subset \mathbb{R}^n$ if $\text{supp}(A) \subset X$. Let $\mathcal{A}_k(U)$ be the vector space of Dirac k -chains supported in U and $\mathcal{B}_k(U)$ the Fréchet space of differential k -forms defined on $U \subseteq \mathbb{R}^n$ each with uniform bounds on each of its directional derivatives. Then $(\mathcal{A}_k(U), \mathcal{B}_k(U))$ is a dual pair. In this paper, we define a family of norms $\|\cdot\|_{B^r}$ on $\mathcal{A}_k(U)$ so that the resulting Banach spaces $\hat{\mathcal{B}}_k^r(U) := (\mathcal{A}_k(U), \|\cdot\|_{B^r})$ form an inverse system. The inductive limit $\hat{\mathcal{B}}_k(U) := \varinjlim \hat{\mathcal{B}}_k^r(U)$ is endowed with the inductive limit topology, which turns out to be the Mackey topology $\tau_k(\mathcal{A}_k(U), \mathcal{B}_k(U))$ on $\mathcal{A}_k(U)$ (see [HP11]). We prove that $\hat{\mathcal{B}}_k(U)$ a well-defined Hausdorff locally convex topological vector space, indeed a (DF) -space (see [HP11]). Note that $(\hat{\mathcal{B}}_k(U))' = \mathcal{B}_k(U)$, by definition of the Mackey topology. We call elements of elements of $\hat{\mathcal{B}}_k(U)$ “differential k -chains of class B in U .”

Remarks 1.1.1.

- The support of a nonzero differential chain $J \in \hat{\mathcal{B}}_k(U)$ is a nonempty closed subset $\text{supp}(J)$ of U (see Theorems 5.0.7 and 6.4.5).
- $\hat{\mathcal{B}}_k(\mathbb{R}^n)$ is a proper subspace of $\mathcal{B}'_k(\mathbb{R}^n)$ (see [HP11]).
- Boundary ∂ and d are continuous on $\hat{\mathcal{B}}_k(U)$ and $\mathcal{B}_k(U)$, respectively, and are in duality (see Theorems 3.5.1 and 6.7.2), so the topological isomorphism $\mathcal{B}_k(U) \cong (\hat{\mathcal{B}}_k(U))'$ (see Theorem 6.2.5) passes to the de Rham isomorphism of cohomology.
- The strong dual topology $\beta(\mathcal{B}_k(U), \hat{\mathcal{B}}_k(U))$ on $\mathcal{B}_k(U)$ coincides with its Fréchet space topology according to [HP11] (see Theorem 2.11.12 for a recap.).
- In §2 and §6.1 we shall provide an explicit construction of the topology on $\hat{\mathcal{B}}_k(U)$ (see pp. 14 and 39).
- In a sequel [Har12a], we show that there exists a continuous coproduct predual to wedge product, as well as a continuous convolution product in $\hat{\mathcal{B}}(M)$ for Lie groups M .

1.2 Integration

We denote the bilinear form $\hat{\mathcal{B}}_k(U) \times \mathcal{B}_k(U) \rightarrow \mathbb{R}$ as the *integral* $f_J \omega := \omega(J)$. We say that a differential k -chain J *represents* a classical domain D of integration if the integrals agree: $f_J \omega = \int_D \omega$ for all $\omega \in \mathcal{B}_k(U)$, where the integral on the right hand side is the Riemann integral. Examples of domains that permit differential chain representatives include open sets (Theorem 2.9.4), polyhedral k -chains, and submanifolds (Proposition 2.10.2), vector fields and foliations (Theorem 8.5.3), and fractals [Har99, Har98]. Roughly speaking, *polyhedral k -chains* are finite sums of weighted oriented simplices with algebraic cancellation and addition of the weights wherever the simplices overlap (see §2.10.3 for a definition). “Curvy” versions of polyhedral k -chains, called “algebraic k -chains” include representatives of all compact k -dimensional Lipschitz immersed submanifolds, and Whitney stratified sets (see §7.1). We can also represent branched k -dimensional generalized surfaces such as soap films, bubbles, crystals, lightning (see §8.4.1 and [Har04b]), given a suitable notion of classical integration over such objects.

1.3 Operator algebras

If X and Y are locally convex, let $\mathcal{L}(X, Y)$ be the space of continuous linear maps from X to Y , and $\mathcal{L}(X)$ the space of continuous operators on X . Three “primitive operators” in $\mathcal{L}(\hat{\mathcal{B}}(U))$ are introduced in §3 and §8. Each is determined by a suitable⁴ vector field V . On simple k -elements, these operators are defined as follows: *extrusion* $E_V(p; \alpha) := (p; V(p) \wedge \alpha)$, *retraction* $E_V^\dagger(p; v_1 \wedge \cdots \wedge v_k) := \sum_{i=1}^k (-1)^{i+1} \langle V(p), v_i \rangle v_1 \wedge \cdots \hat{v}_i \cdots \wedge v_k$ and *prederivative* (first defined in [Har04a] for constant vector fields) $P_V(J) = \lim_{t \rightarrow 0} \phi_{t*}(J/t) - J/t$, where ϕ_t is the time t flow of V , and ϕ_* is the appropriate pushforward. We call $P_V(p; \alpha)$ a *k-element of order one*. Whereas $(p; \alpha)$ is a higher dimensional version of a Dirac delta distribution, $P_V(p; \alpha)$ corresponds to the (negative of the⁵) weak derivative of a Dirac delta. We call such elements *k-elements of order 1 at p*. We may recursively define *k-elements of order s at p* by applying P_V to *k-elements of order s - 1* (see §3.4).

Each of these primitive operators on differential chains dualizes to a classical operator on forms. In particular, $i_V \omega = \omega E_V$, $V^\flat \wedge \omega = \omega E_V^\dagger$, and $L_V \omega = \omega P_V$ where i_V is interior product, and L_V is the Lie derivative.

The resulting integral equations

$$\int_{E_V J} \omega = \int_J i_V \omega \quad [\text{Change of dimension I}] \quad (1.1)$$

$$\int_{E_V^\dagger J} \omega = \int_J V^\flat \wedge \omega \quad [\text{Change of dimension II}] \quad (1.2)$$

$$\int_{P_V J} \omega = \int_J L_V \omega \quad [\text{Change of order I}] \quad (1.3)$$

are explored for constant vector fields V in \mathbb{R}^n in §3 and for smooth vector fields in §8.

Note that it is far from obvious that the primitive operators are continuous and closed in the spaces of differential chains $\hat{\mathcal{B}}(U)$, but once this is established, integral relations are simple consequences of topological duality (Theorems 2.8.2 and 8.4.1). The veracity of such equalities depends on subtle relationships between the differentiability class of the vector field V , the differential form ω , and the chain J , as well as their dimension.

The primitive operators generate other operators on chains via composition. Boundary can be written $\partial = \sum_{i=1}^n P_{e_i} E_{e_i}^\dagger$ where $\{e_i\}$ is an orthonormal basis of \mathbb{R}^n . One of the most surprising aspects of a nonzero k -dimensional Dirac chain is that its boundary is well-defined and nonzero if $k \geq 1$. In particular, a vector $v \in \mathbb{R}^n$ has a well-defined infinitesimal boundary in $\hat{\mathcal{B}}_0(U)$. The resulting Stokes’ Theorem 3.5.4

$$\int_{\partial J} \omega = \int_J d\omega$$

for $J \in \hat{\mathcal{B}}_k(U)$ and $\omega \in \mathcal{B}_{k-1}(U)$ therefore yields an infinitesimal version on Dirac chains. “Perpendicular complement” $\perp = \Pi_{i=1}^n (E_{e_i} + E_{e_i}^\dagger)$ dualizes to Hodge-star on \mathcal{B} , and so we can write down (see Theorem 3.6.3 below, which first appeared in [Har06]) an analogue to Stokes’ theorem for Hodge-star:

$$\int_{\perp J} \omega = \int_J \star \omega.$$

⁴Sufficiently smooth, with bounded derivatives

⁵Weak derivative is the “wrong way around” because its definition is motivated by integration by parts, which treats distributions as “generalized functions,” whereas here we treat them as “generalized domains.”

In §6.5 and §6.6, we find two more fundamental operators, namely, *pushforward* F_* and *multiplication by a function* m_f , where $F : U \rightarrow U'$ is a smooth map and $f : U \rightarrow \mathbb{R}$ is a smooth function with bounded derivatives. These dualize to pullback and multiplication by a function, respectively, giving us change of variables (Corollary 6.5.8) and change of density (Theorem 4.1.6). Notably, bump functions are permitted functions for the operator m_f , as is the unit $f \equiv 1$, giving us partitions of unity. Given “uniform enough” bump functions, the partition of unity sum $1 = \sum f_i$ carries over to differential chains in Theorem 4.2.2: $J = \sum (m_{f_i} J)$.

Vector fields can be represented as differential chains (Theorem 8.5.3), using an inner product or a transverse measure, and we may freely apply pushforward to their representatives, even when the map is not a diffeomorphism. The end result will still be a differential chain, although it may no longer represent a vector field.

In §10 we prove two new fundamental theorems for differential chains in a flow. Given a differential k -chain J and the flow ϕ_t of a smooth vector field V , we define a differential k -chain $\{J_t\}_a^b$ satisfying $\partial\{J_t\}_a^b = \{\partial J_t\}_a^b$ (See Figure 14.) Denoting $J_t := \phi_{t*} J$, we have the following relations:

$$\int_{J_b} \omega - \int_{J_a} \omega = \int_{\{J_t\}_a^b} L_V \omega \quad (\text{Fundamental theorem for differential chains in a flow})$$

and its corollary

$$\int_{\{J_t\}_a^b} dL_V \omega = \int_{\partial J_b} \omega - \int_{\partial J_a} \omega \quad (\text{Stokes' theorem for differential chains in a flow,})$$

given a form ω of corresponding dimension. This is followed by Theorem 10.3.4 Differentiating the Integral in the category of differential chains of class \mathcal{B} . An immediate corollary of 10.3.4 and Cartan’s magic formula for differential chains 3.5.3 is a broad generalization of Reynold’s transport theorem 10.3.6.

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2 Differential chains of class \mathcal{B} in \mathbb{R}^n

We begin our work with differential chains of class \mathcal{B} in \mathbb{R}^n until §6 where we begin to consider chains defined in an open set $U \subseteq \mathbb{R}^n$. In §11 we sketch the basic steps for extending the theory to Riemannian manifolds.

Recall the definition of the space $\mathcal{A}_k(\mathbb{R}^n)$ of Dirac k -chains in \mathbb{R}^n given in §1.1. We assume $0 \leq k \leq n$, unless otherwise specified. What follows is our promised explicit definition of the Mackey topology τ_k on $\mathcal{A}_k(\mathbb{R}^n)$.

2.1 Mass norm

Definition 2.1.1. Choose an inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^n . The inner product extends to $\Lambda_k(T\mathbb{R}^n)$ as follows: Let $\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle := \det(\langle u_i, v_j \rangle)$. (We sometimes use the notation $\langle \alpha, \beta \rangle_\wedge = \langle \alpha, \beta \rangle$.)

Definition 2.1.2. The **mass** of a simple k -vector $\alpha = v_1 \wedge \cdots \wedge v_k$ is given by $\|\alpha\| := \sqrt{\langle \alpha, \alpha \rangle}$. The mass of a k -vector α is

$$\|\alpha\|_{B^0} = \|\alpha\| := \inf \left\{ \sum_{j=1}^N \|(\alpha_i)\| : \alpha_i \text{ are simple, } \alpha = \sum_{j=1}^N \alpha_i \right\}.$$

Define the **mass** of a k -element $(p; \alpha)$ by $\|(p; \alpha)\|_{B^0} := \|\alpha\|_{B^0}$. The **mass** of a Dirac k -chain $A = \sum_{i=1}^m (p_i; \alpha_i)$ is defined by

$$\|A\|_{B^0} := \sum_{i=1}^m \|(p_i; \alpha_i)\|_{B^0}.$$

Another choice of inner product leads to an equivalent mass norm for each k .

2.2 Difference chains

Definition 2.2.1. Given a k -element $(p; \alpha) \in \Lambda_k(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$, let $T_u(p; \alpha) := (p + u; \alpha)$ denote **translation** through u , and $\Delta_u(p; \alpha) := (T_u - I)(p; \alpha)$. Let S^j be the j -th **symmetric power** of \mathbb{R}^n . Denote the **symmetric product** in S by \circ . Let $\sigma = \sigma^j = u_1 \circ \cdots \circ u_j \in S^j$ with $u_i \in \mathbb{R}^n, i = 1, \dots, j$. Recursively define $\Delta_{u \circ \sigma^j}(p; \alpha) := (T_u - Id)(\Delta_{\sigma^j}(p; \alpha))$. The order of the vectors u_i is immaterial since $\Delta_{u \circ v} = \Delta_{v \circ u}$. Let $\|\sigma^j\| := \|u_1\| \cdots \|u_j\|$ and $|\Delta_{\sigma^j}(p; \alpha)|_{B^j} := \|\sigma^j\| \|\alpha\|$. Define $\Delta_{\sigma^0}(p; \alpha) := (p; \alpha)$, to keep the notation consistent. Finally, when there is no ambiguity, we sometimes use the abbreviated notation $\Delta_{\sigma^j} = \Delta_{\sigma^j}(p; \alpha)$. We call $\Delta_{\sigma^j}(p; \alpha)$ a **j -difference k -chain**.

The geometric interpretation of $\Delta_{\sigma^j}(p; \alpha)$ is as follows: $\Delta_{\sigma^j}(p; \alpha)$ is the Dirac chain supported on the vertices of a j -dimensional parallelepiped (possibly degenerate, if the u_i are linearly dependent), with a copy of α at each vertex. The sign of α alternates, but note that $\Delta_{\sigma^j}(p; \alpha) \neq 0$, as long as $\alpha \neq 0$ and $\sigma \neq 0$.

2.3 Norms on Dirac chains

Definition 2.3.1. For $A \in \mathcal{A}_k(\mathbb{R}^n)$ and $r \geq 0$, define the norm

$$\|A\|_{B^r} := \inf \left\{ \sum_{i=1}^m \|\sigma_i^{j(i)}\| \|\alpha_i\| : A = \sum_{i=1}^m \Delta_{\sigma_i^{j(i)}}(p_i; \alpha_i), 0 \leq j(i) \leq r \right\}$$

where each $\sigma_i^{j(i)} \in S^{j(i)}(\mathbb{R}^n)$, $p_i \in \mathbb{R}^n$, $\alpha_i \in \Lambda_k(\mathbb{R}^n)$, and m is arbitrary. That is, we are taking the infimum over all ways to write A as a sum of difference chains, of “order” up to r .

Remarks 2.3.2.

- Although it is clear that $\|\cdot\|_{B^r}$ is a semi-norm, it is not obvious that it is a norm on $\mathcal{A}_k(\mathbb{R}^n)$; this is proved in Theorem 2.6.1.
- It is immediate that the norms are decreasing as r increases, and $\|A\|_{B^r} < \infty$ for all $A \in \mathcal{A}_k(\mathbb{R}^n)$ since $\|A\|_{B^r} \leq \|A\|_{B^0} < \infty$.

- It is not important to know the actual value of $\|A\|_{B^r}$ for a given chain A . Well-behaved upper bounds suffice in our proofs and examples, and these are not usually hard to find. The norms depend only superficially on the choice of inner product since different choices lead to comparable norms, and thus to topologically isomorphic spaces.
- Many of the results in this paper hold for $r = 1$ where the forms are Lipschitz and the chains are of class B^1 . Some readers might be most interested in $0 \leq r \leq 1$ since it takes so little to define the norms, and both smooth and nonsmooth chains can be found, while others might find the inductive limit more useful since its operator algebra is so rich.

2.4 Differential forms of class B^r

An element $\omega \in (\mathcal{A}_k(\mathbb{R}^n))^*$ acts on Dirac chains, and thus can be treated as an exterior k -form. In this section we assume that ω is bounded and measurable.

Definition 2.4.1. *The support of a differential form ω is given by*

$$\text{supp}(\omega) := \overline{\{p \in \mathbb{R}^n : \omega(p; \alpha) \neq 0 \text{ for some } \alpha \in \Lambda_k\}}.$$

Definition 2.4.2. *Define*

$$|\omega|_{B^j} := \sup\{|\omega(\Delta_{\sigma^j}(p; \alpha))| : \|\sigma\| \|\alpha\| = 1\} \text{ and } \|\omega\|_{B^r} = \max\{|\omega|_{B^0}, \dots, |\omega|_{B^r}\}.$$

We say that $\omega \in (\mathcal{A}_k(\mathbb{R}^n))^*$ is a differential form of class B^r if $\|\omega\|_{B^r} < \infty$. Denote the space of differential k -forms of class B^r by $\mathcal{B}_k^r(\mathbb{R}^n)$. Then $\|\cdot\|_{B^r}$ is a norm on $\mathcal{B}_k^r(\mathbb{R}^n)$. (It is straightforward to see that $|\omega|_{B^j}$ satisfies the triangle inequality and homogeneity for each $j \geq 0$. If $\omega \neq 0$, there exists $(p; \alpha)$ such that $\omega(p; \alpha) \neq 0$. Therefore $|\omega|_{B^0} > 0$, and hence $\|\omega\|_{B^r} > 0$.) In §2.8 we show that $\mathcal{B}_k^r(\mathbb{R}^n)$ is topologically isomorphic to $(\mathcal{A}_k(\mathbb{R}^n), \|\cdot\|_{B^r})'$.

Lemma 2.4.3. *Let $A \in \mathcal{A}_k(\mathbb{R}^n)$ and $\omega \in (\mathcal{A}_k(\mathbb{R}^n))^*$ a differential k -form. Then $|\omega(A)| \leq \|\omega\|_{B^r} \|A\|_{B^r}$ for all $r \geq 0$.*

Proof. Let $A \in \mathcal{A}_k(\mathbb{R}^n)$ and $\epsilon > 0$. By the definition of $\|A\|_{B^r}$, there exist $\sigma_i^{j(i)}$ with $j(i) \in \{0, \dots, r\}$ and k -elements $(p_i; \alpha_i)$, $i = 1, \dots, m$, such that $A = \sum_{i=1}^m \Delta_{\sigma_i^{j(i)}}(p_i; \alpha_i)$ and $\|A\|_{B^r} > \sum_{i=1}^m \|\sigma_i^{j(i)}\| \|\alpha_i\| - \epsilon$. The result follows since

$$|\omega(A)| \leq \sum_{i=1}^m |\omega(\Delta_{\sigma_i^{j(i)}})| \leq \sum_{i=1}^m |\omega|_{B^{j(i)}} \|\sigma_i^{j(i)}\| \|\alpha_i\| \leq \|\omega\|_{B^r} (\|A\|_{B^r} + \epsilon).$$

□

2.5 Differential forms of class C^r and $C^{r-1+Lip}$

Definition 2.5.1. *If ω is r -times differentiable, let*

$$|\omega|_{C^j} := \sup \left\{ \frac{|L_{\sigma^j} \omega(p; \alpha)|}{\|\sigma\| \|\alpha\|} \right\}, 0 \leq j \leq r,$$

and $\|\omega\|_{C^r} := \max\{|\omega|_{C^0}, |\omega|_{C^1}, \dots, |\omega|_{C^r}\}$, where L_{σ^j} denotes the j -th directional derivative in the directions u_1, \dots, u_j .

Lemma 2.5.2. *If ω is an exterior form with $|\omega|_{C^r} < \infty$, then $|\omega(\Delta_{\sigma^r}(p; \alpha))| \leq \|\sigma\| \|\alpha\| |\omega|_{C^r}$ for all r -difference k -chains $\Delta_{\sigma^r}(p; \alpha)$ where α is simple and $r \geq 1$.*

Proof. By the Mean Value Theorem, there exists $q = p + su$ such that

$$\frac{\omega(p + tu; \alpha) - \omega(p; \alpha)}{\|u\|} = L_u \omega(q; \alpha).$$

It follows that $|\omega(\Delta_u(p; \alpha))| \leq \|u\| \|\alpha\| |\omega|_{C^1}$.

The proof proceeds by induction. Assume the result holds for r and $|\omega|_{C^{r+1}} < \infty$. Suppose $\sigma = \{u_1, \dots, u_{r+1}\}$, and let $\hat{\sigma} = \{u_1, \dots, u_r\}$. Since $|\omega|_{C^{r+1}} < \infty$, we may apply the mean value theorem again to see that

$$\begin{aligned} |\omega(\Delta_{\sigma^{r+1}}(p; \alpha))| &= |(T_{u_{r+1}}^* \omega - \omega)(\Delta_{\hat{\sigma}^r}(p; 1))| \\ &\leq \|u_1\| \cdots \|u_r\| \|T_{u_{r+1}}^* \omega - \omega\|_{C^r} \\ &\leq \|\sigma\| |\omega|_{C^{r+1}}. \end{aligned}$$

□

Definition 2.5.3. *Let*

$$|\omega|_{Lip} := \sup_{u \neq 0} \left\{ \frac{|\omega(p + u; \alpha) - \omega(p; \alpha)|}{\|u\|} : \|\alpha\| = 1 \right\}.$$

If ω is $(r-1)$ -times differentiable, let

$$|\omega|_{C^{r-1+Lip}} := \sup_{\|u_i\|=1} \{ |L_{u_{r-1}} \circ \cdots \circ L_{u_1} \omega|_{Lip} \},$$

and $\|\omega\|_{C^{r-1+Lip}} := \max\{|\omega|_{C^0}, |\omega|_{C^1}, \dots, |\omega|_{C^{r-1}}, |\omega|_{C^{r-1+Lip}}\}$.

Proposition 2.5.4. *If $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$ is a differential k -form and $r \geq 1$, then ω is r -times differentiable and its r -th order directional derivatives are Lipschitz continuous with $\|\omega\|_{B^r} = \|\omega\|_{C^{r-1+Lip}}$.*

This is a straightforward result in analysis. Details may be found in [Pug09] or earlier versions of this paper posted on the arxiv.

2.6 The B^r norm is indeed a norm on Dirac chains

Theorem 2.6.1. $\|\cdot\|_{B^r}$ *is a norm on Dirac chains $\mathcal{A}_k(\mathbb{R}^n)$ for each $r \geq 0$.*

Proof. Suppose $A \neq 0$ where $A \in \mathcal{A}_k(\mathbb{R}^n)$. It suffices to find nonzero $\omega \in (\mathcal{A}_k(\mathbb{R}^n))^*$ with $\omega(A) \neq 0$ and $\|\omega\|_{B^r} < \infty$. By Lemma 2.4.3 we will then have

$$0 < |\omega(A)| \leq \|\omega\|_{B^r} \|A\|_{B^r} \implies \|A\|_{B^r} > 0.$$

Now $\text{supp}(A) = \{p_0, \dots, p_N\}$. First assume that $A(p_0) = e_I$ for some multi index I . Choose a smooth function f with compact support that is nonzero at p_0 and vanishes at $p_i, 1 \leq i \leq N$. Let

$$\omega_I(p; e_J) = \begin{cases} f(p), & J = I \\ 0, & J \neq I \end{cases}$$

and extend to Dirac chains $\mathcal{A}_k(\mathbb{R}^n)$ by linearity.

Then $\omega_I(A) = \omega_I(p_0; e_I) = 1 \neq 0$. Since f has compact support, $\|f\|_{C^r} < \infty$. We use this to deduce $\|\omega_I\|_{B^r} < \infty$. This reduces to showing $|\omega_I|_{B^j} < \infty$ for each $0 \leq j \leq r$. This, in turn, reduces to showing $|\omega_I(\Delta_{\sigma^j}(p; e_I))| \leq \|\sigma\| \|f\|_{C^j}$. By Lemma 2.5.2 $|\omega_I(\Delta_{\sigma^j}(p; e_I))| = |f(\Delta_{\sigma^j}(p; 1))| \leq \|\sigma\| \|f\|_{C^j} \leq \|\sigma\| \|f\|_{C^r} < \infty$.

In general, $A(p_0) = \sum a_I e_I$ where $a_I \in \mathbb{R}$. Then $\omega = \sum_I \text{sign}(a_I) \omega_I$ satisfies $\|\omega\|_{B^r} < \infty$ and $\omega(A) = \sum_I a_I \text{sign}(a_I) \omega_I(A) = \sum_I |a_I| \neq 0$. \square

Definition 2.6.2. Let $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ be the Banach space obtained upon completion of $\mathcal{A}_k(\mathbb{R}^n)$ with the B^r norm for $r \geq 0$ and $0 \leq k \leq n$. Elements of $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ are called **differential k -chains of class B^r** .

The next result follows from Lemma 2.4.3.

Theorem 2.6.3. The bilinear pairing $\mathcal{B}_k^r(\mathbb{R}^n) \times \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \mathbb{R}$ is separately continuous and satisfies $|\omega(J)| \leq \|\omega\|_{B^r} \|J\|_{B^r}$ for all $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$ and $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$.

2.7 Characterizations of the B^r norms

Lemma 2.7.1. $\|\Delta_v A\|_{B^{r+1}} \leq \|v\| \|A\|_{B^r}$ for all $A \in \mathcal{A}_k(\mathbb{R}^n)$, $v \in \mathbb{R}^n$, and $r \geq 0$.

Proof. Since Dirac chains $\mathcal{A}_k(\mathbb{R}^n)$ are dense in $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, it suffices to prove this for $A \in \mathcal{A}_k(\mathbb{R}^n)$. Let $\epsilon > 0$. We can write $A = \sum_{i=1}^m \Delta_{\sigma_i^{j(i)}}(p_i; \alpha_i)$ as in the proof of Lemma 2.4.3, such that

$$\|A\|_{B^r} > \sum_{i=1}^m \|\sigma_i^{j(i)}\| \|\alpha_i\| - \epsilon.$$

Then

$$\begin{aligned} \|\Delta_v A\|_{B^{r+1}} &= \|T_v A - A\|_{B^{r+1}} \leq \sum_{i=1}^m \|T_v \Delta_{\sigma_i^{j(i)}} - \Delta_{\sigma_i^{j(i)}}\|_{B^{r+1}} \leq \sum_{i=1}^m \|T_v \Delta_{\sigma_i^{j(i)}} - \Delta_{\sigma_i^{j(i)}}\|_{B^{j+1}} \\ &\leq \|v\| \sum_{i=1}^m \|\sigma_i^{j(i)}\| \|\alpha_i\| \\ &< \|v\| (\|A\|_{B^r} + \epsilon). \end{aligned}$$

\square

Lemma 2.7.2. The norm $\|\cdot\|_{B^r}$ is the largest seminorm $|\cdot|'$ on Dirac chains $\mathcal{A}_k(\mathbb{R}^n)$ such that $|\Delta_{\sigma^j}(p; \alpha)|' \leq \|\sigma\| \|\alpha\|$ for all j -difference k -chains $\Delta_{\sigma^j}(p; \alpha)$, $0 \leq j \leq r$.

Proof. First observe that the B^r norm itself satisfies this inequality by its definition. On the other hand, suppose $|\cdot|'$ is a seminorm satisfying $|\Delta_{\sigma^j}(p; \alpha)|' \leq \|\sigma\| \|\alpha\|$. Let $A \in \mathcal{A}_k(\mathbb{R}^n)$ be a Dirac chain and $\epsilon > 0$. We can write $A = \sum_{i=1}^m \Delta_{\sigma_i^{j(i)}}(p_i; \alpha_i)$ as in the proof of Lemma 2.4.3, with $\|A\|_{B^r} > \sum_{i=1}^m \|\sigma_i^{j(i)}\| \|\alpha_i\| - \epsilon$. Therefore, by the triangle inequality, $|A|' \leq \sum_{i=1}^m |\Delta_{\sigma_i^{j(i)}}(p_i; \alpha_i)|' \leq \sum_{i=1}^m \|\sigma_i^{j(i)}\| \|\alpha_i\| < \|A\|_{B^r} + \epsilon$. Since this estimate holds for all $\epsilon > 0$, the result follows. \square

Theorem 2.7.3. *The norm $\|\cdot\|_{B^r}$ is the largest seminorm $|\cdot|'$ on Dirac chains $\mathcal{A}_k(\mathbb{R}^n)$ such that*

- $|A|' \leq \|A\|_{B^0}$
- $|\Delta_u A|' \leq \|u\| \|A\|_{B^{r-1}}$.

for all $r \geq 1$ and $A \in \mathcal{A}$.

Proof. $\|\cdot\|_{B^r}$ satisfies the first part since the B^r norms are decreasing on chains. It satisfies the second inequality by Lemma 2.7.1. On the other hand, suppose $|\cdot|'$ is a seminorm satisfying the two conditions. For $j = 0$, we use the first inequality $|(p; \alpha)|' \leq \|(p; \alpha)\|_{B^0} = \|\alpha\|$. Fix $0 < j \leq r$. Using induction and recalling the notation $\hat{\sigma}$ from Lemma 2.5.2, it follows that

$$|\Delta_{\sigma^j}(p; \alpha)|' = |\Delta_{\hat{\sigma}^{j-1}, u_1}(p; \alpha)|' \leq \|u_1\| \|\Delta_{\hat{\sigma}^{j-1}}(p; \alpha)\|_{B^{j-1}} \leq \|\sigma\| \|\alpha\|.$$

Therefore, the conditions of Lemma 2.7.2 are met and we deduce $|A|' \leq \|A\|_{B^r}$ for all Dirac chains A . \square

Let $\mathcal{A} = \oplus_{k=0}^n \mathcal{A}_k(\mathbb{R}^n)$. Recall $\Delta_u A = T_u A - A$ where $T_u(p; \alpha) = (p + u; \alpha)$.

Corollary 2.7.4. *If $T : \mathcal{A} \rightarrow \mathcal{A}$ is an operator satisfying $\|T(\Delta_{\sigma^j}(p; \alpha))\|_{B^r} \leq C \|\sigma\| \|\alpha\|$ for some constant $C > 0$ and all j -difference k -chains $\Delta_{\sigma^j}(p; \alpha)$ with $0 \leq j \leq s$ and $0 \leq k \leq n$, then $\|T(A)\|_{B^r} \leq C \|A\|_{B^s}$ for all $A \in \mathcal{A}_k(\mathbb{R}^n)$.*

Proof. Let $|A|' = \frac{1}{C} \|T(A)\|_{B^r}$. Then $|A|'$ is a seminorm, and the result follows by Theorem 2.7.3. \square

2.8 Isomorphism of differential forms and differential cochains

In this section we show that the Banach space $\mathcal{B}_k^r(\mathbb{R}^n)$ of differential forms is topologically isomorphic to the Banach space $(\hat{\mathcal{B}}_k^r)'$ of differential cochains.

Theorem 2.8.1. *Let $r \geq 0$. If $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$ is a differential k -form, then $\|\omega\|_{B^r} = \sup_{0 \neq A \in \mathcal{A}_k} \frac{|\omega(A)|}{\|A\|_{B^r}}$.*

Proof. We know $|\omega(A)| \leq \|\omega\|_{B^r} \|A\|_{B^r}$ by Lemma 2.4.3. On the other hand,

$$|\omega|_{B^j} = \sup_{0 \neq \Delta_{\sigma^j}(p; \alpha)} \frac{|\omega(\Delta_{\sigma^j}(p; \alpha))|}{|\Delta_{\sigma^j}(p; \alpha)|_{B^j}} \leq \sup_{0 \neq \Delta_{\sigma^j}(p; \alpha)} \frac{|\omega(\Delta_{\sigma^j}(p; \alpha))|}{\|\Delta_{\sigma^j}(p; \alpha)\|_{B^r}} \leq \sup_{0 \neq A} \frac{|\omega(A)|}{\|A\|_{B^r}}.$$

It follows that $\|\omega\|_{B^r} \leq \sup_{0 \neq A} \frac{|\omega(A)|}{\|A\|_{B^r}}$. \square

Theorem 2.8.2. *The linear map $\Psi_r : (\hat{\mathcal{B}}_k^r(\mathbb{R}^n))' \rightarrow \mathcal{B}_k^r(\mathbb{R}^n)$ determined by $\Psi_r(X)(p; \alpha) := X(p; \alpha)$ is a topological isomorphism for each $r \geq 0$. Furthermore,*

$$\|\Psi_r(X)\|_{B^r} = \|X\|_{B^r}.$$

Proof. Let $X \in (\hat{\mathcal{B}}_k^r(\mathbb{R}^n))'$ be a cochain. We show $\|\Psi_r(X)\|_{B^r} < \|X\|_{B^r}$. Now

$$\frac{|\Psi_r(X)(\Delta_{\sigma^j}(p; \alpha))|}{|\Delta_{\sigma^j}(p; \alpha)|_{B^j}} = \frac{|X(\Delta_{\sigma^j}(p; \alpha))|}{|\Delta_{\sigma^j}(p; \alpha)|_{B^j}} \leq \frac{|X(\Delta_{\sigma^j}(p; \alpha))|}{\|\Delta_{\sigma^j}(p; \alpha)\|_{B^r}} \leq \|X\|_{B^r}$$

Therefore, $|\Psi_r(X)|_{B^j} \leq \|X\|_{B^r}$, and thus $\|\Psi_r(X)\|_{B^r} \leq \|X\|_{B^r}$.

It follows that $\Psi_r : (\hat{\mathcal{B}}_k^r(\mathbb{R}^n))' \rightarrow \mathcal{B}_k^r(\mathbb{R}^n)$ is a continuous linear map. We show Ψ_r is a continuous isomorphism. Let $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$. Let $\Theta_r : \mathcal{B}_k^r(\mathbb{R}^n) \rightarrow (\hat{\mathcal{B}}_k^r(\mathbb{R}^n))'$ be given by $\Theta_r \omega(A) := \omega(A)$. Then $\Theta_r \omega$ is a linear functional on $\mathcal{A}_k(\mathbb{R}^n)$. By Theorem 2.8.1

$$\|\Theta_r \omega\|_{B^r} = \sup_{A \neq 0} \frac{|\Theta_r \omega(A)|}{\|A\|_{B^r}} = \sup_{A \neq 0} \frac{|\omega(A)|}{\|A\|_{B^r}} = \|\omega\|_{B^r}.$$

Therefore, $\Theta_r \omega \in (\hat{\mathcal{B}}_k^r(\mathbb{R}^n))'$. We conclude that Ψ_r is a topological isomorphism with inverse Θ_r . \square

Theorem 2.8.3. *If $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ is a differential chain and $r \geq 0$, then $\|J\|_{B^r} = \sup_{0 \neq \omega \in \mathcal{B}_k^r} \frac{|\omega(J)|}{\|\omega\|_{B^r}}$.*

2.8.1 Integration

We may use any dense subspace to approximate the integral such as Dirac chains, simplicial complexes, and polyhedral chains. Smooth submanifolds, and even fractals, can also be used to approximate the integral since their representatives generate dense subspaces of $\hat{\mathcal{B}}$. This unifies a variety of viewpoints in mathematics.

Although the space $\mathcal{B}_k^r(\mathbb{R}^n)$ of differential forms is dual to $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, and thus we are perfectly justified in using the notation $\omega(J)$, this can become confusing since we are no longer evaluating ω at a set of points, as we were with elements of $\mathcal{A}_k(\mathbb{R}^n)$. Instead we shall think of $\omega(J)$ as “integration” over J and denote it as $\int_J \omega := \omega(J)$. Indeed, the integral notation is justified by the approximation of $\omega(J)$ by its analogue to Riemann sums. That is,

$$\int_J \omega = \lim_{i \rightarrow \infty} \omega(A_i),$$

where $A_i \in \mathcal{A}_k(\mathbb{R}^n)$ and $A_i \rightarrow J$ in $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$. In this new notation, Theorem 2.6.3, for example, becomes

$$\left| \int_J \omega \right| \leq \|J\|_{B^r} \|\omega\|_{B^r}. \quad (2.1)$$

Of course, $\int_A \omega = \omega(A)$ for $A \in \mathcal{A}_k$, and we shall use either notation.

We shall see later in Theorems 2.9.4, 2.10.2, 8.5.3, and 7.1.1 that when J represents a classical domain A of integration such as an open set, polyhedral chain, vector field or submanifold, respectively, the Riemann integral $\int_A \omega$ and $\int_J \omega$ agree⁶.

The Banach space of differential chains $\hat{\mathcal{B}}_k^1(\mathbb{R}^n)$ is separable since the subspace of Dirac chains is dense and has a countable dense subset, but the Banach space of Lipschitz forms $\mathcal{B}_k^1(\mathbb{R}^n)$ is not separable [Whi57].

The next result follows from Theorem 2.8.2.

Corollary 2.8.4. *$\int : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \otimes \mathcal{B}_k^r(\mathbb{R}^n) \rightarrow \mathbb{R}$ is a nondegenerate bilinear pairing for $r \geq 0$.*

⁶This is in fact our definition of the word “represent,” but we construct these chains first, then show the integrals agree. That is, they are not found implicitly.

2.9 Representatives of open sets

We say that an n -chain \tilde{U} represents an open set U , if $\int_{\tilde{U}} \omega = \int_U \omega$ for all forms $\omega \in \mathcal{B}_n^1(\mathbb{R}^n)$, where the integral on the right hand side is the Riemann integral.) We show there exists a unique element \tilde{U} in $\hat{\mathcal{B}}_n^1(\mathbb{R}^n)$ representing U for each bounded, open subset $U \subset \mathbb{R}^n$.

Let Q be an n -cube in \mathbb{R}^n with unit side length. For each $j \geq 1$, subdivide Q into 2^{nj} non-overlapping sub-cubes $Q_{j,i}$, using a binary subdivision. Let $q_{j,i}$ be the midpoint of $Q_{j,i}$ and $P_j = \sum_{i=1}^{2^{nj}} (q_{j,i}; 2^{-nj} \mathbb{1})$ where $\mathbb{1} = e_1 \wedge \cdots \wedge e_n$. Then

Lemma 2.9.1. $\|P_j\|_{B^r} = 1$ for all $r \geq 0$.

Proof. For each $r \geq 0$, $\|P_j\|_{B^r} \leq \sum_{i=1}^{2^{nj}} \|(q_{j,i}; 2^{-nj} \mathbb{1})\|_{B^r} \leq 2^{nj} 2^{-nj} = 1$. By Theorem 2.8.3 $\|P_j\|_{B^r} \geq \int_{P_j} dV = 1$ since $\|dV\|_{B^r} = 1$. \square

Proposition 2.9.2. The sequence $\{P_j\}$ is Cauchy in the B^1 norm.

Proof. Rather than write out some inequalities with messy subscripts, we shall describe what is going on and why the result is true. We would like to estimate $\|P_j - P_{j+1}\|_{B^1}$. Both Dirac n -chains P_j and P_{j+1} have mass one. They are supported in sets of points placed at midpoints of the binary grids with side length 2^{-j} and $2^{-(j+1)}$, respectively. Each of the n -elements of P_j has mass 2^{-nj} and those of P_{j+1} have mass $2^{-n(j+1)}$. The key idea is to think of each simple n -element of P_j as 2^n duplicate copies of simple n -elements of mass $2^{-n(j+1)}$ supported at the same point. This gives us a 1–1 correspondence between the simple n -elements of P_j and those of P_{j+1} . We can choose a bijection so that the distance between points paired is less than 2^{-j+1} . Use the triangle inequality with respect to $P_j - P_{j+1}$ written as a sum of differences of these paired n -elements and Lemma 2.7.1 to obtain $\|P_j - P_{j+1}\|_{B^1} \leq 2^{-j+1}$. It follows that $\{P_j\}$ is Cauchy in the B^1 norm since the series $\sum 2^{-j+1}$ converges. \square

Denote the limit $\tilde{Q} := \lim_{j \rightarrow \infty} P_j$ in the B^1 norm. Then $\tilde{Q} \in \hat{\mathcal{B}}_n^1(\mathbb{R}^n)$ is a well-defined differential n -chain. By continuity of the differential chain integral (2.1) and the definition of the Riemann integral as a limit of Riemann sums, we have

$$\int_{\tilde{Q}} \omega = \lim_{j \rightarrow \infty} \int_{\sum (p_{j,i}; \alpha_{j,i})} \omega = \int_Q \omega.$$

That is, \tilde{Q} represents Q . If $W = \cup_i Q_i$ is a finite union of non-overlapping cubes, then $\tilde{W} := \sum \tilde{Q}_i$ represents U .

Definition 2.9.3. The **frontier** of an open set U is defined by $\text{fr}(U) := \overline{U} \setminus U$.

We shall reserve the word “boundary” for the operator on differential chains developed in §3.5, as well as for the classical boundary of a submanifold, both of which have orientation. Frontier is defined for sets, with no algebraic properties. Thus when we speak of the boundary of an open set, we are thinking of U as a submanifold, and when we speak of its frontier, we are thinking of U as a set.

Theorem 2.9.4. Let $U \subset \mathbb{R}^n$ be bounded and open. There exists a unique differential n -chain $\tilde{U} \in \hat{\mathcal{B}}_n^1(\mathbb{R}^n)$ such that $\int_{\tilde{U}} \omega = \int_U \omega$ for all $\omega \in \mathcal{B}_n^1(\mathbb{R}^n)$, where the integral on the right hand side is the Riemann integral. Furthermore, $\|\tilde{U}\|_{B^1} = \int_U dV$, the volume of U .

Proof. Let $U = \cup_{i=1}^{\infty} Q_i$ be a Whitney decomposition of U into a union of non-overlapping n -cubes, and $W_s = \sum_{i=1}^s \widetilde{Q}_i$. We first show that the differential chains \widetilde{W}_s form a Cauchy sequence in $\hat{\mathcal{B}}_n^1(\mathbb{R}^n)$. Now $\|\widetilde{W}_t - \widetilde{W}_s\|_{B^1} = \sum_{i=s+1}^t \|\widetilde{Q}_i\|_{B^1} \leq \sum_{i=s+1}^t \|\widetilde{Q}_i\|_{B^0} = \sum_{i=s+1}^t \int_{Q_i} dV$ tends to zero as $s \leq t \rightarrow \infty$ since the last sum is bounded by the volume of a small neighborhood of the frontier of U , a bounded open set. Therefore $\sum_{i=1}^{\infty} \widetilde{Q}_i$ converges to a well-defined chain $\widetilde{U} \in \hat{\mathcal{B}}_n^1(\mathbb{R}^n)$. That is,

$$\widetilde{U} = \sum_{i=1}^{\infty} \widetilde{Q}_i \quad (2.2)$$

where convergence is in the B^1 norm. Suppose $\omega \in \mathcal{B}_n^1(\mathbb{R}^n)$. Then $f_{\widetilde{U}} \omega = \lim_{s \rightarrow \infty} f_{\sum_{i=1}^s \widetilde{Q}_i} \omega = \int_U \omega$ by the definition of the Riemann integral.

We first prove the last assertion for positively oriented unit n -cubes $Q \subset \mathbb{R}^n$. Recall from Proposition 2.9.2 that $\widetilde{Q} = \lim_{j \rightarrow \infty} P_j$ where the $P_j \in \mathcal{A}_n(\mathbb{R}^n)$. Using Lemma 2.9.1, we have $\|\widetilde{Q}\|_{B^1} = \lim_{j \rightarrow \infty} \|P_j\|_{B^1} = 1 = \int_Q dV$. The result follows for cubes with arbitrary side length by linearity. Now let U be any bounded and open subset of \mathbb{R}^n . By (2.2) we know $\|\widetilde{U}\|_{B^1} = \lim_{s \rightarrow \infty} \|\sum_{i=1}^s \widetilde{Q}_i\|_{B^1} \leq \lim_{s \rightarrow \infty} \sum_{i=1}^s \|\widetilde{Q}_i\|_{B^1} = \lim_{s \rightarrow \infty} \sum_{i=1}^s \int_{Q_i} dV = \int_U dV$. Finally, since $dV \in \mathcal{B}_n^1(\mathbb{R}^n)$ and $\|dV\|_{B^1} = 1$ we use Theorem 2.8.3 to obtain $\|\widetilde{U}\|_{B^1} \geq f_{\widetilde{U}} dV = \int_U dV$. \square

2.10 Polyhedral chains

We can similarly represent affine k -cells in \mathbb{R}^n , and using these, define *polyhedral chains*:

Definition 2.10.1. Recall that an **affine n -cell** in \mathbb{R}^n is the intersection of finitely many affine half spaces in \mathbb{R}^n whose closure is compact. The half spaces may be open or closed. An affine n -cell can be partly open and partly closed. That is, it is the union of an open n -cell with possibly some of its faces. An **affine k -cell** in \mathbb{R}^n is an affine k -cell in a k -dimensional affine subspace of \mathbb{R}^n .

Theorem 2.10.2. If σ is an oriented affine k -cell in \mathbb{R}^n , there is a unique differential k -chain $\widetilde{\sigma} \in \hat{\mathcal{B}}_k^1(\mathbb{R}^n)$ such that

$$\int_{\widetilde{\sigma}} \omega = \int_{\sigma} \omega$$

for all $\omega \in \mathcal{B}_k^1(\mathbb{R}^n)$, where the right hand integral is the Riemann integral.

Proof. The result follows from Theorem 2.9.4 applied to the k -dimensional subspace of \mathbb{R}^n containing σ . \square

Definition 2.10.3. A **polyhedral k -chain** in U^7 is a finite sum $\sum_{i=1}^s a_i \widetilde{\sigma}_i$ where $a_i \in \mathbb{R}$ and $\widetilde{\sigma}_i \in \hat{\mathcal{B}}_k^1(\mathbb{R}^n)$ represents an oriented affine k -cell σ_i in U .

In §7.1 we use pushforward to create smooth versions of polyhedral chains, resulting in representatives of compact submanifolds.

⁷An equivalent definition uses simplices instead of cells. Every simplex is a cell and every cell can be subdivided into finitely many simplices

2.10.1 Polyhedral chains are dense in differential chains

Let $p \in \mathbb{R}^n$ and e_I a basis element of Λ_k . Let Q_j be a k -cube centered at p , with side length 2^{-j} , and contained in the affine plane containing p and parallel to the k -direction of e_I . Orient Q_k to match the orientation of e_I .

Lemma 2.10.4. $(p; e_I) = \lim_{j \rightarrow \infty} 2^{jk} \tilde{Q}_j$ in the B^1 norm.

Proof. By Theorem 2.9.4 the cube Q_j is represented by $\tilde{Q}_j \in \hat{\mathcal{B}}_k^1(\mathbb{R}^n)$. We know from Proposition 2.9.2 that \tilde{Q}_j is the limit of Dirac k -chains $A_{j,i}$ supported in the midpoints of the $2^{-(j+i)}$ binary subdivision of Q_j . The total mass of $A_{j,i}$ is 2^{-jk} and each of the k -elements has k -direction and orientation the same as that of e_I . We can translate each of the simple k -elements of $A_{j,i}$ to p . The distance translated is less than 2^{-j} . The triangle inequality shows that $\|2^{jk} A_{j,i} - (p; e_I)\|_{B^1} \leq 2^{-j}$. The result follows since $\|2^{jk} \tilde{Q}_j - (p; e_I)\|_{B^1} \leq \|2^{jk} \tilde{Q}_j - 2^{jk} A_{j,i}\|_{B^1} + \|2^{jk} A_{j,i} - (p; e_I)\|_{B^1}$. \square

It follows that if $\alpha \in \Lambda_k$ is simple, then $(p; \alpha)$ is the limit of renormalized representatives of k -cubes in the B^1 norm by writing $\alpha = \sum a_I e_I$.

Theorem 2.10.5. Polyhedral k -chains are dense in the Banach space of differential k -chains $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ for all $r \geq 1$.

Proof. This follows from Lemma 2.10.4 and since Dirac chains are dense. \square

Remark 2.10.6. Lemma 2.10.4 does not rely on any particular shape, or open set to approximate $(p; \alpha)$. It certainly does not need to be a cube, nor do we have to use a sequence of homothetic replicas. We may use a sequence of chains whose supports tend to p and whose k -vectors tend to α . See [Har12b].

Example 2.10.7. We show how to represent the middle third Cantor set using a sequence of polyhedral 1-chains. Let E_1 be the chain representing the oriented interval $(0, 1)$. Let C_1 represent $(1/3, 2/3)$, and let $E_2 = E_1 + (-C_1)$. We have replaced the word “erase” with the algebraically precise “subtract” (see p.10). Recursively define E_n by subtracting the middle third of E_{n-1} . The mass of E_n is $(\frac{2}{3})^n$. It is not hard to show that the sequence $\{(\frac{2}{3})^n E_n\}$ forms a Cauchy sequence in $\hat{\mathcal{B}}_1^1(\mathbb{R}^n)$. Therefore its limit is a differential 1-chain Γ in $\hat{\mathcal{B}}_1^1(\mathbb{R}^n)$. (See §2.10.7 where the boundary operator is applied to Γ .)

2.11 The inductive limit topology

Since the B^r norms are decreasing, the identity map $(\mathcal{A}_k(\mathbb{R}^n), \|\cdot\|_{B^r}) \rightarrow \hat{\mathcal{B}}_k^s(\mathbb{R}^n)$ is continuous and linear whenever $r \leq s$, and therefore extends to the completion, $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$. The resulting continuous linear maps $u_k^{r,s} : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^s(\mathbb{R}^n)$ are well-defined **linking maps**. The next lemma is straightforward:

Lemma 2.11.1. The maps $u_k^{r,s} : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^s(\mathbb{R}^n)$ satisfy

- (a) $u_k^{r,r} = I$;
- (b) $u_k^{s,t} \circ u_k^{r,s} = u_k^{r,t}$ for all $r \leq s \leq t$;
- (c) The image $u_k^{r,s}(\hat{\mathcal{B}}_k^r(\mathbb{R}^n))$ is dense in $\hat{\mathcal{B}}_k^s(\mathbb{R}^n)$.

In Corollary 2.11.4 below we will show that each $u_k^{r,s} : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^s(\mathbb{R}^n)$ is an injection. This will follow from knowing that forms of class B^r are approximated by forms of class B^{r+1} in the B^r norm. Whitney proved this for C^r forms using convolution product (see [Whi57] Chapter V, Theorem 13A), and our proof is similar.

For $c > 0$, let $\overline{\kappa_c} : [0, \infty) \rightarrow \mathbb{R}$ be a nonnegative smooth function that is monotone decreasing, constant on some interval $[0, t_0]$ and equals 0 for $t \geq c$. Let $\kappa_c : \mathbb{R}^n \rightarrow \mathbb{R}$ be given by $\kappa_c(v) = \overline{\kappa_c}(\|v\|)$. Let $dV = dx_1 \wedge \cdots \wedge dx_n$ the unit n -form. We multiply κ_c by a constant so that $\int_{\mathbb{R}^n} \kappa_c(v) dV = 1$. Let a_r be the volume of an n -ball of radius r in \mathbb{R}^n . For $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$ a differential k -form and $(p; \alpha)$ a simple k -element, let

$$\omega_c(p; \alpha) = \int_{\mathbb{R}^n} \kappa_c(v) \omega(p + v; \alpha) dV.$$

Theorem 2.11.2. *If $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$ and $c > 0$, then $\omega_c \in \mathcal{B}_k(\mathbb{R}^n)$ and*

- (a) $L_u(\omega_c) = (L_u\omega)_c$ for all $u \in \mathbb{R}^n$;
- (b) $\|\omega_c\|_{B^r} \leq \|\omega\|_{B^r}$;
- (c) $\omega_c \in \mathcal{B}_k^{r+1}(\mathbb{R}^n)$;
- (d) $\omega_c(J) \rightarrow \omega(J)$ as $c \rightarrow 0$ for all $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$.

Proof. (a):

$$\begin{aligned} L_u(\omega_c)(p; \alpha) &= \lim_{\epsilon \rightarrow 0} \omega_c(p + \epsilon u; \alpha) - (p; \alpha) / \epsilon = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} \kappa_c(v) \omega(p + \epsilon u + v; \alpha) - (p + v; \alpha) / \epsilon dV \\ &= \int_{\mathbb{R}^n} \kappa_c(v) (L_u \omega(p + v; \alpha)) dV = (L_u \omega)_c(p; \alpha). \end{aligned}$$

(b): Since $\int_{\mathbb{R}^n} \kappa_c dV = 1$, we know

$$\begin{aligned} \frac{|\omega_c(\Delta_{\sigma^j}(p; \alpha))|}{\|\sigma\| \|\alpha\|} &= \left| \int_{\mathbb{R}^n} \kappa_c(v) \omega(T_v \Delta_{\sigma^j}(p; \alpha) / \|\sigma\| \|\alpha\|) dV \right| \leq \sup_{v \neq 0} \frac{|\omega(T_v \Delta_{\sigma^j}(p; \alpha))|}{\|\sigma\| \|\alpha\|} \\ &= \sup_{v \neq 0} \frac{|\omega(\Delta_{\sigma^j}(p + v; \alpha))|}{\|\sigma\| \|\alpha\|} \leq \sup_{q \in \mathbb{R}^n} \frac{|\omega(\Delta_{\sigma^j}(q; \alpha))|}{\|\Delta_{\sigma^j}(q; \alpha)\|_{B^r}} \leq \sup_{0 \neq A \in \mathcal{A}_k} \frac{|A|}{\|A\|_{B^r}} = \|\omega\|_{B^r} \end{aligned}$$

for all $0 \leq j \leq r$. Therefore, $|\omega_c|_{B^j} \leq \|\omega\|_{B^r}$, and hence $\|\omega_c\|_{B^r} \leq \|\omega\|_{B^r}$.

(c): Suppose $0 \leq j \leq r$. Let $\eta = L_{\sigma^j} \omega$. Then $|\eta|_{B^0} \leq \|\omega\|_{B^r} < \infty$. By (a) we know $\eta_c = L_{\sigma^j}(\omega_c)$. Now

$$\eta_c(T_u(p; \alpha)) = \int_{\mathbb{R}^n} \kappa_c(v) \eta(T_{v+u}(p; \alpha)) dV = \int_{\mathbb{R}^n} \kappa_c(v - u) \eta(T_v(p; \alpha)) dV,$$

and

$$\eta_c(p; \alpha) = \int_{\mathbb{R}^n} \kappa_c(v) \eta(T_v(p; \alpha)) dV.$$

Since the integrand vanishes for v outside ball of radius c about the origin, we have

$$\begin{aligned} |\eta_c(p+u; \alpha) - \eta_c(p; \alpha)| &= \left| \int_{\mathbb{R}^n} (\kappa_c(v-u) - \kappa_c(v)) \eta(p+v; \alpha) dV \right| \\ &\leq \int_{\mathbb{R}^n} |\kappa_c(v-u) - \kappa_c(v)| |\eta(p+v; \alpha)| dV \\ &\leq a_c |\kappa_c|_{Lip} \|u\| \|\eta\|_{B^0} \|\alpha\|. \end{aligned}$$

Therefore, $|\eta_c|_{Lip} \leq a_c |\kappa|_{Lip} \|\omega\|_{B^r}$. Using (b), Proposition 2.5.4 and Theorem 2.8.1, we deduce $\|\omega_c\|_{B^{r+1}} < \infty$.

(d): First of all

$$(\omega_c - \omega)(p; \alpha) = \int_{\mathbb{R}^n} \kappa_c(v) \|v\| \omega((p+v; \alpha) - (p; \alpha)) / \|v\| dV = \int_{\mathbb{R}^n} \kappa_c(v) \|v\| L_v \omega(p; \alpha) dV + r_c$$

where $r_c \rightarrow 0$. Let A be a Dirac chain. Then

$$|(\omega_c - \omega)(A)| \leq c \left| \int_{\mathbb{R}^n} \kappa_c(v) L_v \omega(A) dV \right| \leq c \left| \sup_v \{\kappa_c(v)\} \int_{\mathbb{R}^n} L_v \omega(A) dV \right| \leq c \sup_v \{\kappa_c(v)\} a_c |\omega(A)| \leq c' \|\omega\|_{B^r} \|A\|_{B^r}$$

where $c' \rightarrow 0$ as $c \rightarrow 0$. For $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ choose $A_i \rightarrow J$ in $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ and use $\|A_i\|_{B^r} \rightarrow \|J\|_{B^r}$. \square

Corollary 2.11.3. *If $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ is a differential k -chain and $0 \leq r$, then*

$$\|J\|_{B^r} = \sup_{0 \neq \omega \in \mathcal{B}_k^{r+1}} \frac{|\omega(J)|}{\|\omega\|_{B^r}}.$$

Proof. According to Theorem 2.8.3, we have $\|J\|_{B^r} = \sup_{0 \neq \omega \in \mathcal{B}_k^r} \frac{|\omega(J)|}{\|\omega\|_{B^r}}$. Suppose $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$. Let $\epsilon > 0$. By Theorem 2.11.2 (b)-(d), there exists $c > 0$ such that

$$\frac{|\omega(J)|}{\|\omega\|_{B^r}} \leq \frac{|\omega_c(J)| + \epsilon}{\|\omega\|_{B^r}} \leq \frac{|\omega_c(J)| + \epsilon}{\|\omega_c\|_{B^r}} \leq \sup_{\eta \in \mathcal{B}_k^{r+1}} \frac{|\eta(J)| + \epsilon}{\|\eta\|_{B^r}}.$$

Since this holds for all $\epsilon > 0$, we know $\|J\|_{B^r} = \sup_{\omega \in \mathcal{B}_k^r(\mathbb{R}^n)} \frac{|\omega(J)|}{\|\omega\|_{B^r}} \leq \sup_{\eta \in \mathcal{B}_k^{r+1}(\mathbb{R}^n)} \frac{|\eta(J)|}{\|\eta\|_{B^r}}$. Equality holds since $\mathcal{B}^{r+1} \subset \mathcal{B}^r$. \square

Corollary 2.11.4. *The linking maps $u_k^{r,s} : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \hookrightarrow \hat{\mathcal{B}}_k^s(\mathbb{R}^n)$ are injections for each $r \leq s$.*

Proof. It suffices to prove this for $s = r+1$. Suppose there exists $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ with $u_k^{r,r+1}(J) = 0$. Then $\omega(u_k^{r,r+1}(J)) = 0$ for all $\omega \in \mathcal{B}_k^{r+1}(\mathbb{R}^n)$. By Corollary 2.11.3, this implies that $\|J\|_{B^r} = 0$, and hence $J = 0$. \square

Definition 2.11.5. *Let $\hat{\mathcal{B}}_k(\mathbb{R}^n) = \hat{\mathcal{B}}_k^\infty(\mathbb{R}^n) := \varinjlim \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, the inductive limit as $r \rightarrow \infty$, and $u_k^r : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k(\mathbb{R}^n)$ the canonical inclusion maps into the inductive limit.*

It is sometimes the case that results in $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ carry over easily to the inductive limit $\hat{\mathcal{B}}_k(\mathbb{R}^n)$, but not always⁸. For example, is the inductive limit Hausdorff? Is it complete? Is it reflexive? Is it Montel? Does every bounded subset of the inductive limit come from a bounded subset of one of the Banach spaces? Much of [HP11] is devoted to answering questions of this type, as well as the current paper.

Corollary 2.11.6. *The canonical inclusions $u_k^r : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k(\mathbb{R}^n)$ are injective.*

Proof. We know from Corollary 2.11.4 that the linking maps are injective. If the linking maps $u_k^{r,s}$ defining an inductive limit of vector spaces are injective, then so are the canonical inclusions u_k^r (see [Köt66], p. 219). \square

Each $u_k^r(\hat{\mathcal{B}}_k^r(\mathbb{R}^n))$ is a Banach space with norm $\|u_k^r(J)\|_r = \|J\|_{B^r}$. Since the B^r norms are decreasing, then $\{u_k^r(\hat{\mathcal{B}}_k^r(\mathbb{R}^n))\}$ forms a nested sequence of increasing Banach subspaces of $\hat{\mathcal{B}}_k(\mathbb{R}^n)$. We endow $\hat{\mathcal{B}}_k(\mathbb{R}^n)$ with the inductive limit topology τ_k ; it is the finest locally convex topology such that the maps $u_k^r : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k(\mathbb{R}^n)$ are continuous. If F is locally convex, a linear map $T : (\hat{\mathcal{B}}_k(\mathbb{R}^n), \tau_k) \rightarrow F$ is continuous if and only if each $T \circ u_k^r : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow F$ is continuous⁹. However, the inductive limit is not strict. For example, k -dimensional dipoles are limits of chains in $u_0(\hat{\mathcal{B}}_k^0(\mathbb{R}^n))$ in $(\hat{\mathcal{B}}_k(\mathbb{R}^n), \tau_k)$, but dipoles are not found in the Banach space $u_0(\hat{\mathcal{B}}_k^0(\mathbb{R}^n))$ itself.

Definition 2.11.7. *If $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n) = \varinjlim \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, the **type** of J is the infimum over all $r > 0$ such that there exists $J^r \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ with $u_k^r(J^r) = J$.*

Type is well-defined since the Banach spaces $u_k^r(\hat{\mathcal{B}}_k^r(\mathbb{R}^n))$ generate $\hat{\mathcal{B}}_k(\mathbb{R}^n)$ (see [Gro73], p. 136).

Theorem 2.11.8. *The locally convex space $(\hat{\mathcal{B}}_k(\mathbb{R}^n), \tau_k)$ is Hausdorff and separable.*

Proof. Suppose $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$ is nonzero. Since the class of J is well-defined, there exists $J^r \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ with $u_k^r(J^r) = J$ where $u_k^r : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k(\mathbb{R}^n)$ is the canonical injection. Thus $J^r \neq 0$. By Theorem 2.11.2 there exists $\omega \in B^\infty$ with $\omega(J^r) \neq 0$. It follows that $\omega(J) \neq 0$ since J and J^r are approximated by the same Cauchy sequence of Dirac chains. Therefore $\hat{\mathcal{B}}_k(\mathbb{R}^n)$ is separated. This implies that $\hat{\mathcal{B}}_k(\mathbb{R}^n)$ is Hausdorff (see [Gro73] p. 59).

For the second part, since Dirac chains are dense in each $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, we can approximate a given Dirac chain D by a sequence of Dirac chains $D_i = \sum_j (p_{j,i}; \alpha_{j,i})$ whose points $p_{j,i}$ and k -vectors $\alpha_{j,i}$ have rational coordinates. \square

Definition 2.11.9. *Let $\mathcal{B}_k(\mathbb{R}^n)$ denote the Fréchet space of bounded C^∞ -smooth differential k -forms, with bounds on the j -th order directional derivatives for each $0 \leq j < \infty$. The defining seminorms can be taken to be the B^r norms.*

⁸Grothendieck wrote, “Some questions arise concerning a space which is an inductive limit, which often receive negative answers, even for the inductive limits of a sequence of Banach spaces, and which often present serious difficulties. ” and “We remark that in practice the difficulties which we encounter in inductive limits are the ‘converse’ of those met in projective limits (the coarsest topology for which...); here it is nearly always easy to show that the space is complete, and to determine whether its bounded subsets are weakly compact or compact..., an in particular to recognize it as either a reflexive or a Montel space.” See [Gro73], p 138.

⁹One can also introduce Hölder conditions into the classes of differential chains and forms as follows: For $0 < \beta \leq 1$, replace $|\Delta_{\sigma^j}(p; \alpha)|_{B^j} = \|u_j\| \cdots \|u_1\| \|\alpha\|$ in definition (2.3.1) with $|\Delta_{\sigma^j}(p; \alpha)|_{B^{j-1+\beta}} = \|u_j\|^\beta \cdots \|u_1\| \|\alpha\|$ where $\sigma^j = u_j \circ \cdots \circ u_1$. The resulting spaces of chains $\hat{\mathcal{B}}_k^{r-1+\beta}(\mathbb{R}^n)$ can permit fine tuning of the class of a chain. The inductive limit of the resulting spaces is the same $\hat{\mathcal{B}}_k(\mathbb{R}^n)$ as before, but we can now define the *extrinsic dimension* of a chain $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$ as $\dim_E(J) := \inf\{k+j-1+\beta : J \in \hat{\mathcal{B}}_k^{j-1+\beta}(\mathbb{R}^n)\}$. For example, $\dim_E(\tilde{\sigma}_k) = k$ where σ_k is an affine k -cell since $\tilde{\sigma}_k \in \hat{\mathcal{B}}_k^1(\mathbb{R}^n)$ (set $j = 0, \beta = 1$), while $\dim_E(\tilde{S}) = \ln(3)/\ln(2)$ where S is the Sierpinski triangle. The dual spaces are differential forms of class $B^{r-1+\beta}$, i.e., the forms are of class B^{r-1} and the $(r-1)$ order directional derivatives satisfy a β Hölder condition. The author’s earliest work on this theory focused more on Hölder conditions, but she has largely set this aside in recent years.

Proposition 2.11.10. *The topological dual to $\hat{\mathcal{B}}_k(\mathbb{R}^n)$ is isomorphic to $\mathcal{B}_k(\mathbb{R}^n)$.*

Proof. It is well known that the dual of the inductive limit topology is the projective limit of the duals, and vice versa. (See [Köt66], §22.7, for example.). Since $\hat{\mathcal{B}}_k(\mathbb{R}^n) = \varinjlim \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, we know $(\hat{\mathcal{B}}_k(\mathbb{R}^n))' = \varprojlim (\hat{\mathcal{B}}_k^r(\mathbb{R}^n))' \cong \varprojlim \mathcal{B}_k^r(\mathbb{R}^n) = \mathcal{B}_k(\mathbb{R}^n)$. \square

Let F be the Fréchet topology on the space of differential k -forms $\mathcal{B}_k(\mathbb{R}^n)$ and $\beta(\mathcal{B}_k(\mathbb{R}^n), \hat{\mathcal{B}}_k(\mathbb{R}^n))$ the strong topology of the dual pair $(\mathcal{B}_k(\mathbb{R}^n), \hat{\mathcal{B}}_k(\mathbb{R}^n))$.

The following theorem was first established in [HP11]:

Theorem 2.11.11. $(\mathcal{B}_k(\mathbb{R}^n), \beta(\mathcal{B}_k(\mathbb{R}^n), \hat{\mathcal{B}}_k(\mathbb{R}^n))) = (\mathcal{B}_k(\mathbb{R}^n), F)$.

We immediately deduce the fundamental topological isomorphism of cochains and forms by J. Harrison and H. Pugh in [HP11], without which results in this paper involving the inductive limit $\hat{\mathcal{B}}(\mathbb{R}^n)$ would have little substance.

Theorem 2.11.12. *The space of differential k -cochains $(\hat{\mathcal{B}}_k(\mathbb{R}^n))'$ with the strong (polar) topology is topologically isomorphic to the Fréchet space of differential k -forms $\mathcal{B}_k(\mathbb{R}^n)$.*

Our integral pairing (see Corollary 2.8.4) therefore extends to a separately continuous pairing $f_J \omega := \omega(J)$ for $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$ and $\omega \in \mathcal{B}_k(\mathbb{R}^n)$.

We next describe an equivalent way to construct the inductive limit $(\hat{\mathcal{B}}_k(\mathbb{R}^n), \tau_k)$ is via direct sums and quotients due to K othe [Köt66]. This is the approach we shall use for it provides a clear way to establish continuity of operators in the inductive limit, and is in line with the algebraic viewpoint in [Har13].

Definition 2.11.13. *For $0 \leq k \leq n$, let $H_k = H_k(\mathbb{R}^n)$ be the **hull** of the linking maps, that is, the linear span of the subset $\{J^r - u_k^{r,s}(J^r) : J^r \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n), s \geq r \geq 0\} \subset \oplus_r \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$.*

First endow $\oplus_{r=0}^\infty \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ with the direct sum topology. Then $\oplus_r \hat{\mathcal{B}}_k^r(\mathbb{R}^n)/H_k$, endowed with the quotient topology, is topologically isomorphic to $(\hat{\mathcal{B}}_k(\mathbb{R}^n), \tau_k)$ ([Köt66], §19.2, p.219). Since $(\hat{\mathcal{B}}_k(\mathbb{R}^n), \tau_k)$ is Hausdorff (Theorem 2.11.8), it follows that H_k is closed. ([Köt66], (4) p.216). Hence the projection $\pi_k : \oplus_r \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \oplus_r \hat{\mathcal{B}}_k^r(\mathbb{R}^n)/H_k$ is a continuous linear map ([Köt66], §10.7 (3),(4)). Since the canonical inclusion $\nu_k^r : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \oplus_r \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ is continuous, the inclusion $u_k^r := \pi_k \circ \nu_k^r : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow (\hat{\mathcal{B}}_k(\mathbb{R}^n), \tau_k)$ is continuous.

Suppose $T : \oplus_{k=0}^n \oplus_{r \geq 0} \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \oplus_{k=0}^n \oplus_{r \geq 0} \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ is continuous and bigraded with $T(\hat{\mathcal{B}}_k^r(\mathbb{R}^n)) \subset \hat{\mathcal{B}}_\ell^s(\mathbb{R}^n)$, and $T(H_k) \subset H_\ell$. By the universal property of quotients, T factors through a graded linear map $\hat{T} : \oplus_k \oplus_r \hat{\mathcal{B}}_k^r(\mathbb{R}^n)/H_k \rightarrow \oplus_k \oplus_r \hat{\mathcal{B}}_k^r(\mathbb{R}^n)/H_k$ with $T = \hat{T} \circ \pi$ where π is the canonical projection onto the quotient. That is, $\hat{T}[H_k + J] = [H_k + T(J)]$. Let $\hat{\mathcal{B}}(\mathbb{R}^n) = \hat{\mathcal{B}}^\infty(\mathbb{R}^n) := \oplus_{k=0}^n \hat{\mathcal{B}}_k(\mathbb{R}^n)$, endowed with the direct sum topology.

Theorem 2.11.14. *If $T : \oplus_{k=0}^n \oplus_{r=0}^\infty \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \oplus_{k=0}^n \oplus_{r=0}^\infty \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ is a continuous bigraded linear map with $T(\hat{\mathcal{B}}_k^r(\mathbb{R}^n)) \subset \hat{\mathcal{B}}_\ell^s(\mathbb{R}^n)$, and $T(H_k(\mathbb{R}^n)) \subset H_\ell(\mathbb{R}^n)$, then T factors through a continuous graded linear map $\hat{T} : \hat{\mathcal{B}}(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}(\mathbb{R}^n)$ with $T = \pi \circ \hat{T}$.*

Proof. By the universal property of quotients, T factors through a graded linear map $\hat{T} : \oplus_k \oplus_r \hat{\mathcal{B}}_k^r(\mathbb{R}^n)/H_k \rightarrow \oplus_k \oplus_r \hat{\mathcal{B}}_k^r(\mathbb{R}^n)/H_k$ with $T = \hat{T} \circ \pi$ where π is the canonical projection onto the quotient. That is, $\hat{T}[H_k + J] = [H_k + T(J)]$. Furthermore, $\hat{T} : \hat{\mathcal{B}}(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}(\mathbb{R}^n)$ is continuous and graded (see [Köt66] §19.1 (7), p. 217). \square

A dual result holds for projective limits and can be found in [Köt66] (§19.6 (6), p. 227), replacing locally convex hulls with locally convex kernels and reversing all the arrows, but we do not need such formality.

3 Primitive operators

In this section we define a few simple operators on the space $\hat{\mathcal{B}}(\mathbb{R}^n)$ of differential chains. The “primitive” operators are all defined initially with respect to a vector $v \in \mathbb{R}^n$ (or perhaps more formally but no more rigorously, a constant vector field determined by v .) In §8.2.2, we extend them to bona fide vector fields, and in §6.9, we extend them again to differential chains on an open set U .

3.1 Extrusion

Definition 3.1.1. Let $E : \mathbb{R}^n \times \mathcal{A}_k(\mathbb{R}^n) \rightarrow \mathcal{A}_{k+1}(\mathbb{R}^n)$ be the bilinear map, called **extrusion**, defined by its action on simple k -elements,

$$E(v, (p; \alpha)) := (p; v \wedge \alpha).$$

Let $E_v(p; \alpha) := E(v, (p; \alpha))$.

Lemma 3.1.2. For each $r \geq 0$ and $0 \leq k \leq n$, the map $E_v : \mathcal{A}_k(\mathbb{R}^n) \rightarrow \mathcal{A}_{k+1}(\mathbb{R}^n)$ extends to a continuous linear map $E_v : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_{k+1}^r(\mathbb{R}^n)$ satisfying $\|E_v(J)\|_{B^r} \leq \|v\| \|J\|_{B^r}$ for all $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$.

Proof. Let $T(p; \alpha) = E_u(p; \alpha)$ where $u \in \mathbb{R}^n$ is unit, and $0 \leq j \leq r$. Then

$$\|T(\Delta_{\sigma^j}(p; \alpha))\|_{B^r} = \|\Delta_{\sigma^j} T(p; \alpha)\|_{B^r} = \|\Delta_{\sigma^j}(p; u \wedge \alpha)\|_{B^r} \leq \|\sigma\| \|\alpha\|.$$

By Corollary 2.7.4 it follows that $\|E_v(A)\|_{B^r} \leq \|v\| \|A\|_{B^r}$. The extension to $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ is immediate. \square

If S and T are operators, let $[S, T] = ST - TS$ and $\{S, T\} = ST + TS$. The following relations are immediate:

Proposition 3.1.3. Let $v, w \in \mathbb{R}^n$. Then

$$(a) \ E_v^2 = 0;$$

$$(b) \ \{E_v, E_w\} = 0.$$

Definition 3.1.4. Let $E_\alpha : \mathcal{A}_k(\mathbb{R}^n) \rightarrow \mathcal{A}_{k+\ell}(\mathbb{R}^n)$ be the linear map defined by $E_\alpha := E_{v_\ell} \circ \cdots \circ E_{v_1}$ where $\alpha = v_1 \wedge \cdots \wedge v_\ell$ is simple.

Then $E_\alpha(p; \beta) = (p; \alpha \wedge \beta)$. The inequality in Lemma 3.1.2 readily extends to $\|E_\alpha(J)\|_{B^r} \leq \|\alpha\| \|J\|_{B^r}$.

Since $E_v(H_k) \subset H_{k+1}$ we know from Theorem 2.11.14 that $E_v : \oplus_{k=0}^n \oplus_{r=0}^\infty \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \oplus_{k=0}^n \oplus_{r=0}^\infty \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ factors through a continuous graded linear map $\hat{E}_v : \hat{\mathcal{B}}(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}(\mathbb{R}^n)$ with $\hat{E}_v(\hat{\mathcal{B}}_k^r(\mathbb{R}^n)) \subset \hat{\mathcal{B}}_{k+1}^r(\mathbb{R}^n)$. To keep the notation simple, we will suppress the quotient and write E_v instead of \hat{E}_v .

The dual operator on B is *interior product* $i_v : \mathcal{B}_{k+1}^r(\mathbb{R}^n) \rightarrow \mathcal{B}_k^r(\mathbb{R}^n)$ since i_v satisfies $\omega E_v(p; \alpha) = i_v \omega(p; \alpha)$.

Corollary 3.1.5. Let $v \in \mathbb{R}^n$ and $0 \leq k \leq n-1$. Then $E_v \in \mathcal{L}(\hat{\mathcal{B}})$ and $i_v \in \mathcal{L}(\mathcal{B})$ are continuous graded operators satisfying

$$\int_{E_v J} \omega = \int_J i_v \omega \tag{3.1}$$

for all matching pairs $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, $\omega \in \mathcal{B}_{k+1}^r(\mathbb{R}^n)$, and $0 \leq r \leq \infty$.

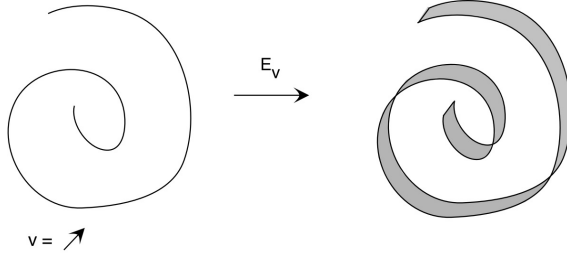


Figure 1: Extrusion of a curve through the vector field $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

Proof. This follows from the isomorphism theorems 2.8.1 and 2.11.12. □

Remarks 3.1.6.

- Only a few operators we work with are closed on Dirac chains. When this occurs, as it does for extrusion E_v , the corresponding integral relation given by duality is still nontrivial, because continuity must be shown, and because of the isomorphism theorems 2.8.1 and 2.11.12.
- This result will be significantly extended in §8.2 when we replace $v \in \mathbb{R}^n$ with a vector field in a Fréchet space of smooth vector fields
- Separate continuity will be proved in §8.2.

3.2 Retraction

Given a k -vector $\alpha = v_1 \wedge \cdots \wedge v_k$, $k \geq 1$, let $\hat{\alpha}_i = v_1 \wedge \cdots \wedge \hat{v}_i \cdots \wedge v_k$. For $k = 1$, set $\hat{\alpha} = 0$.

Definition 3.2.1. Let $E^\dagger : \mathbb{R}^n \times \mathcal{A}_k(\mathbb{R}^n) \rightarrow \mathcal{A}_{k-1}(\mathbb{R}^n)$ be the bilinear map, called **retraction**, defined by its action on simple k -elements,

$$E^\dagger(v, (p; \alpha)) := \sum_{i=1}^k (-1)^{i+1} \langle v, v_i \rangle (p; \hat{\alpha}_i).$$

Let $E_v^\dagger(p; \alpha) := E^\dagger(v, (p; \alpha))$.

Theorem 3.2.2. For each $r \geq 0$ and $0 \leq k \leq n$, the map E_v^\dagger extends to a continuous linear map $E_v^\dagger : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_{k-1}^r(\mathbb{R}^n)$ satisfying

- (a) $\|E_v^\dagger(J)\|_{B^r} \leq \binom{n}{k} k \|v\| \|J\|_{B^r}$ for all $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, and
- (b) $E_v^\dagger(H_k) \subset H_{k-1}$.

Proof. (a): Let $u \in \mathbb{R}^n$ be a unit vector and $e_I = e_1 \wedge \cdots \wedge e_k$. Then

$$\|E_u^\dagger(\Delta_{\sigma^j}(p; e_I))\|_{B^r} \leq \sum_{i=1}^k |(-1)^{i+1} \langle u, e_i \rangle \Delta_{\sigma^j}(p; \hat{e}_{I_i})|_{B^j} \leq k \|\sigma\|.$$

Suppose $\alpha = \sum_I a_I e_I$ and $v \in \mathbb{R}^n$. It follows that

$$\|E_v^\dagger(\Delta_{\sigma^j}(p; \alpha))\|_{B^r} \leq \sum_I |a_I| \|E_v^\dagger(\Delta_{\sigma^j}(p; e_I))\|_{B^r} \leq k \|v\| \sum_I |a_I| \|\sigma\| \leq \binom{n}{k} k \|v\| \|\alpha\| \|\sigma\|.$$

Now apply Corollary 2.7.4 to obtain the inequality (a). Part (b) follows since $u_{k-1}^{r-1, r} E_v^\dagger = E_v^\dagger u_k^{r-1, r}$. \square

The extension to continuous graded linear maps $E_v^\dagger : \hat{\mathcal{B}}(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}(\mathbb{R}^n)$ is similar to our construction for extrusion E_v and we omit the details.

There is no need for us to keep track of the constant showing up in (a), only that it depends only on k and n , and not r . We do not see a coordinate-free way to establish (a) because the definition of E_v^\dagger heavily relies on the inner product.

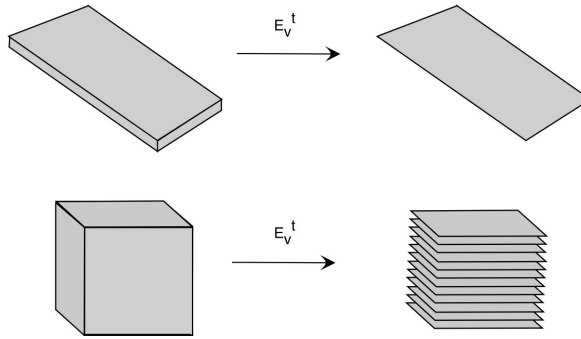


Figure 2: Retraction of an extruded rectangle and a unit cube

Recall the inner product $\langle \cdot, \cdot \rangle_\wedge$ in §2.1.

Proposition 3.2.3. *The following relations hold:*

- (a) $\langle E_v(p; \alpha), (p; \beta) \rangle_\wedge = \langle (p; \alpha), E_v^\dagger(p; \beta) \rangle_\wedge$ for all k -vectors β and $(k-1)$ -vectors α ;
- (b) $E_v^\dagger \circ E_v^\dagger = 0$ for all $v \in \mathbb{R}^n$;
- (c) $[E_v^\dagger, E_w^\dagger] = 0$ for all $v, w \in \mathbb{R}^n$;
- (d) $\{E_v^\dagger, E_w\} = \{E_v, E_w^\dagger\} = \langle v, w \rangle I$;
- (e) $(E_v + E_v^\dagger)^2 = \langle v, v \rangle I$;

Proof. Property (a) comes from the well-established result that E_v^\dagger is the adjoint to E_v as operators on the exterior algebra. (These are the creation and annihilation operators of the exterior algebra as in the study of Fock spaces, e.g., [Att].) The relation can be verified directly by writing everything in terms of an orthonormal basis $\{e_1, \dots, e_n\}$. Here is the rough idea: We may assume $\alpha = e_I = e_1 \wedge \dots \wedge e_{k-1}$ and $v = e_k$. For if $v = e_i, 1 \leq i \leq k-1$, then both

sides are zero. We may also assume $\beta = e_k \wedge e_I$, else both sides are again zero. But if $\beta = e_k \wedge e_I$, then both sides are one. The general result follows using bilinearity of the inner product.

The remaining properties follow directly from the definitions. \square

Property (d) coincides with the canonical anticommutation relations (C.A.R.) of Fock spaces (see [Att]).

Lemma 3.2.4. *The dual operator of E_v^\dagger is $v^\flat \wedge \cdot$.*

Proof. It is enough to show that $\omega E_v^\dagger(p; \alpha) = v^\flat \wedge \omega(p; \alpha)$ for all k -elements $(p; \alpha)$ and all $v \in \mathbb{R}^n$. This follows since

$$\omega E_v^\dagger(p; \alpha) = \sum_{i=1}^n (-1)^{i+1} \omega(p; \langle v, v_i \rangle \alpha_i) = v^\flat \wedge \omega(p; \alpha).$$

\square

Corollary 3.2.5. *Let $v \in \mathbb{R}^n$ and $1 \leq k \leq n$. Then*

$$\int_{E_v^\dagger J} \omega = \int_J v^\flat \wedge \omega \quad (3.2)$$

for all matching pairs $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, $\omega \in \mathcal{B}_{k-1}^r(\mathbb{R}^n)$, and $0 \leq r \leq \infty$.

Proof. This follows from the isomorphism theorems 2.8.1 and 2.11.12. \square

3.3 Prederivative

In this section we introduce the concept of differentiation of chains, without regard to a function.

Lemma 3.3.1. *Let $(p; \alpha)$ be a simple k -element and $u \in \mathbb{R}^n$. Then the sequence $\{(T_{2^{-i}u} - I)(p; 2^i \alpha)\}_{i \geq 1}$ is Cauchy in $\hat{\mathcal{B}}_k^2(\mathbb{R}^n)$.*

Proof. For simplicity of notation, set $p = 0$. Telescoping gives us

$$(2^{-i}u; \alpha) - (0; \alpha) = \sum_{m=1}^{2^j} (m2^{-i+j}u; \alpha) - ((m-1)2^{-i+j}u; \alpha). \quad (3.3)$$

Expanding, we have

$$(T_{2^{-i}u} - Id)(0; 2^i \alpha) - (T_{2^{-(i+j)}u} - Id)(0; 2^{i+j} \alpha) = ((2^{-i}u; 2^i \alpha) - (0; 2^i \alpha)) - ((2^{-(i+j)}u; 2^{i+j} \alpha) - (0; 2^{i+j} \alpha)).$$

We apply (3.3) to the first pair, and consider the second pair as 2^j copies of $((2^{-(i+j)}u; 2^i \alpha) - (0; 2^i \alpha))$. Then

$$\begin{aligned} & (T_{2^{-i}u} - Id)(0; 2^i \alpha) - (T_{2^{-(i+j)}u} - Id)(0; 2^{i+j} \alpha) \\ &= \sum_{m=1}^{2^j} (m2^{-(i+j)}u; 2^i \alpha) - ((m-1)2^{-(i+j)}u; 2^i \alpha) - ((2^{-(i+j)}u; 2^i \alpha) - (0; 2^i \alpha)). \end{aligned}$$

Rewrite the right hand side as a sum of 2-difference chains and set $v = 2^{-(i+j)}u$. Then

$$(T_{2^{-i}u} - Id)(0; 2^i \alpha) - (T_{2^{-(i+j)}u} - Id)(0; 2^{i+j} \alpha) = \sum_{m=1}^{2^j} \Delta_{(v, (m-1)v)}(0; 2^i \alpha). \quad (3.4)$$

We then use the triangle inequality to deduce

$$\begin{aligned} \|(T_{2^{-i}u} - Id)(0; 2^i \alpha) - (T_{2^{-(i+j)}u} - Id)(0; 2^{i+j} \alpha)\|_{B^2} &\leq \sum_{m=1}^{2^j} \|\Delta_{(v, (m-1)v)}(0; 2^i \alpha)\|_{B^2} \\ &\leq \sum_{m=1}^{2^j} |\Delta_{(v, (m-1)v)}(0; 2^i \alpha)|_{B^2} \leq 2^{-i} M(\alpha). \end{aligned}$$

□

Definition 3.3.2. Define the differential chain $P_v(p; \alpha) := \lim_{i \rightarrow \infty} (T_{iv} - I)_*(p; \alpha/i)$ and extend to a linear map of Dirac chains $P_v : \mathcal{A}_k(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^2(\mathbb{R}^n)$ by linearity. Let $P : \mathbb{R}^n \times \mathcal{A}_k(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^2(\mathbb{R}^n)$ be the resulting bilinear map: $P(v, (p; \alpha)) := P_v(p; \alpha)$ called **prederivative**. Using the natural inclusion $u_k^{2,r} : \hat{\mathcal{B}}_k^2(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, we have $u_k^{2,r} \circ P_v : \mathcal{A}_k(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, but we usually write $P_v : \mathcal{A}_k(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ instead of $u_k^{2,r} \circ P_v : \mathcal{A}_k(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$.

Lemma 3.3.3. For each $v \in \mathbb{R}^n$ and $r \geq 1$, the map $P_v : \mathcal{A}_k(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^{r+1}(\mathbb{R}^n)$ satisfies

- (a) $\|P_v(A)\|_{B^{r+1}} \leq \|v\| \|A\|_{B^r}$ for all $A \in \mathcal{A}_k(\mathbb{R}^n)$;
- (b) $P_v(H_k) \subset H_k$.

Proof. (a): Since P_v commutes with translation, and P_v is defined as a limit of 1-difference chains, we have

$$\|P_v(\Delta_{\sigma^j}(p; \alpha))\|_{B^{j+1}} = \lim_{t \rightarrow 0} \|\Delta_{tv}(\Delta_{\sigma^j}(p; \alpha/t))\|_{B^{j+1}} \leq \lim_{t \rightarrow 0} |\Delta_{tv}(\Delta_{\sigma^j}(p; \alpha/t))|_{B^{j+1}} \leq \|v\| \|\sigma\| \|\alpha\|.$$

Let $A \in \mathcal{A}_k(\mathbb{R}^n)$ be a Dirac chain, $r \geq 0$, and $\epsilon > 0$. We can write $A = \sum_{i=1}^m \Delta_{\sigma_i^{j(i)}}(p_i; \alpha_i)$ as in the proof of Lemma 2.4.3, with $\|A\|_{B^r} > \sum_{i=1}^m \|\sigma_i\| \|\alpha_i\| - \epsilon$. Then

$$\|P_v A\|_{B^{r+1}} \leq \sum_{i=1}^m \|P_v \Delta_{\sigma_i^{j(i)}}(p_i; \alpha_i)\|_{B^{r+1}} \leq \|v\| \sum_{i=1}^m \|\sigma_i\| \|\alpha_i\| < \|v\| (\|A\|_{B^r} + \epsilon).$$

- (b): This follows since $P_v \circ u_k^{r,s} = u_k^{r,s} \circ P_v$.

□

Definition 3.3.4. Define $P_v : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^{r+1}$ as follows: If $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, choose $A_i \rightarrow J$ in the B^r norm and define $P_v(J) := \lim_{i \rightarrow \infty} P_v(A_i)$.

We deduce

Proposition 3.3.5. The map $P : \mathbb{R}^n \times \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^{r+1}(\mathbb{R}^n)$ is bilinear and satisfies $\|P_v(J)\|_{B^{r+1}} \leq \|v\| \|J\|_{B^r}$ for all $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$.

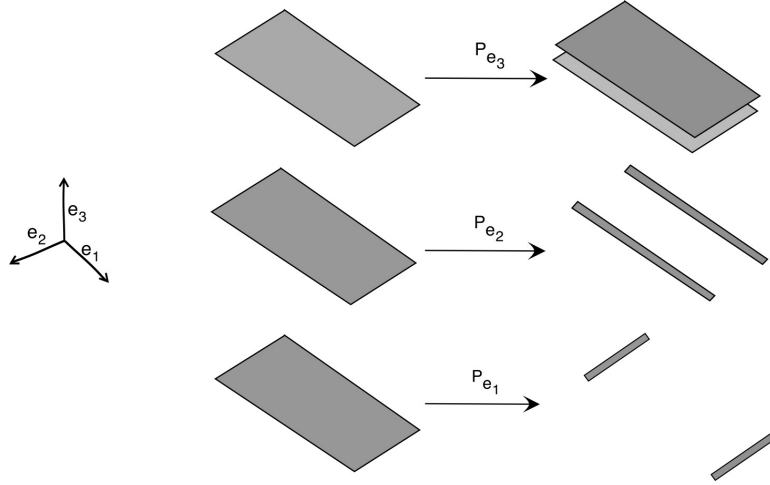


Figure 3: Prederivative of a 2-cell in \mathbb{R}^3 in three different directions

Proof. The inequality holds by Lemma 3.3.3. We show that P determines a bilinear map on Dirac chains: It is linear in the second variable by definition. Additivity in the first variable reduces to showing

$$\lim_{t \rightarrow 0} (p + t(v_1 + v_2); \alpha/t) - (p + tv_1; \alpha/t) - (p + tv_2; \alpha/t) + (p; \alpha/t) = 0$$

in $\hat{\mathcal{B}}_k(\mathbb{R}^n)$. This follows from the definition of the B^2 norm of a 2-difference chain: $\|(p + t(v_1 + v_2); \alpha/t) - (p + tv_1; \alpha/t) - (p + tv_2; \alpha/t) + (p; \alpha/t)\|_{B^2} \leq |(p + t(v_1 + v_2); \alpha/t) - (p + tv_1; \alpha/t) - (p + tv_2; \alpha/t) + (p; \alpha/t)|_2 \leq t\|v_1\|\|v_2\|\|\alpha\|$. Homogeneity is immediate since $\lambda(p; \alpha) = (p; \lambda\alpha)$. \square

In particular, Theorem 3.3.5 shows that $P_v \in \mathcal{L}(\hat{\mathcal{B}})$ is a continuous bigraded operator.

Proposition 3.3.6 (Commutation relations). $[P_v, P_w] = [E_v, P_w] = [E_v^\dagger, P_w] = 0$ for all $v, w \in \mathbb{R}^n$.

Lemma 3.3.7. The dual operator to P_v is directional¹⁰ derivative $L_v \omega(p; \alpha) := \lim_{h \rightarrow 0} \frac{\omega(p + tv; \alpha) - \omega(p; \alpha)}{t}$.

Proof. We have $\omega P_v(p; \alpha) = \omega \lim_{t \rightarrow 0} \frac{(p + tv; \alpha) - (p; \alpha)}{t} = \lim_{t \rightarrow 0} \frac{\omega(p + tv; \alpha) - \omega(p; \alpha)}{t} = L_v \omega(p; \alpha)$. \square

Note that when we exchange the limit, we may do so because ω is a continuous operator on differential chains. Differentiability here is translated into continuity.

Corollary 3.3.8. Let $v \in \mathbb{R}^n$ and $0 \leq k \leq n$. Then $P_v \in \mathcal{L}(\hat{\mathcal{B}})$ and $L_v \in \mathcal{L}(\mathcal{B})$ are continuous bigraded operators satisfying

$$\int_{P_v J} \omega = \int_J L_v \omega \quad (3.5)$$

for all matching pairs $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, $\omega \in \mathcal{B}_k^{r+1}(\mathbb{R}^n)$, and $1 \leq r \leq \infty$.

¹⁰when the derivative is with respect to a bona fide vector field, we will call it the Lie derivative.

3.4 Chainlets

Let $\sigma = \sigma^s = \{u_1, \dots, u_s\}$ be a list of vectors in \mathbb{R}^n and $P_{\sigma^s}(p; \alpha) = P_{u_1} \circ \dots \circ P_{u_s}(p; \alpha)$ where $(p; \alpha)$ is a simple k -element. We say p is the *support*¹¹ of $P_{\sigma^s}(p; \alpha)$. Then the differential chain $P_{\sigma^s}(p; \alpha) \in \hat{\mathcal{B}}_k^{s+1}(\mathbb{R}^n)$ is called a *simple k -element of order s* . For $k = 0$, these are *singular distributions* represented geometrically as differential chains, e.g., Dirac deltas ($s = 0$), dipoles ($s = 1$), and quadrupoles ($s = 2$). For $s \geq 0$, let $\mathcal{A}_k^s(p)$ be the subspace of $\hat{\mathcal{B}}_k^{s+1}(\mathbb{R}^n)$ generated by k -elements of order s supported at p . Elements of $\mathcal{A}_k^s(p)$ are called *Dirac k -chains of order s supported at p* . Let $S(\mathbb{R}^n)$ be the symmetric algebra and $S^s(\mathbb{R}^n)$ its s -th symmetric power.

Proposition 3.4.1. *Let $p \in \mathbb{R}^n$. The vector space $\mathcal{A}_k^s(p)$ is isomorphic to $S^s(\mathbb{R}^n) \otimes \Lambda_k(\mathbb{R}^n)$.*

Proof. Let $k = 0$. The linear map $P_{u_s} \circ \dots \circ P_{u_1}(p; \alpha) \mapsto u_s \otimes \dots \otimes u_1 \otimes \alpha$ is an isomorphism preserving the symmetry of both sides since $P_{u_1} \circ P_{u_2} = P_{u_2} \circ P_{u_1}$ and $P_{u_1} \circ (P_{u_2} \circ P_{u_3}) = (P_{u_1} \circ P_{u_2}) \circ P_{u_3}$. \square

Remark 3.4.2. *We may therefore use the notation $(p; u \otimes \alpha) = P_u(p; \alpha)$, and, more generally, $(p; \sigma \otimes \alpha) = (p; (u_s \circ \dots \circ u_1) \otimes \alpha) = P_{u_s} \circ \dots \circ P_{u_1}(p; \alpha) = P_\sigma(p; \alpha)$ where $\sigma = u_s \circ \dots \circ u_1$. For example, $P_u(p; \sigma \otimes \alpha) = (p; (u \circ \sigma) \otimes \alpha)$. We will be using both notations, preferring whichever one fits the situation best. While P_σ emphasizes the operator viewpoint, the tensor product notation reveals the algebra of the Koszul complex $\bigoplus_{s=0}^\infty \bigoplus_{k=0}^n S^s \otimes \Lambda_k$ more clearly. For example, the scalar t in $t(\sigma \otimes \alpha) = t\sigma \otimes \alpha = \sigma \otimes t\alpha$ “floats,” while this is not immediately clear using the operator notation.*

Definition 3.4.3 (Unital associative algebra at p). *Define $\mathcal{A}_\bullet^s(p) := \bigoplus_{k=0}^n \bigoplus_{s \geq 0} \mathcal{A}_k^s(p)$. Let \cdot be the product on higher order Dirac chains $\mathcal{A}_\bullet^s(p)$ given by $(p; \sigma \otimes \alpha) \cdot (p; \tau \otimes \beta) := (p; \sigma \circ \tau \otimes \alpha \wedge \beta)$.*

Remarks 3.4.4.

- For each $p \in \mathbb{R}^n$ $\mathcal{A}_\bullet^s(p)$ is a unital, associative algebra with unit $(p; 1)$.
- This product is not continuous and does not extend to the topological vector space $\hat{\mathcal{B}}(\mathbb{R}^n)$.
- E_v^\dagger is a graded derivation on the algebra $\mathcal{A}_\bullet^s(p)$. Neither E_v nor P_v are derivations.

Lemma 3.4.5. *There exists a unique bigraded operator $\partial : \mathcal{A}_k^s(p) \rightarrow \mathcal{A}_{k-1}^{s+1}(p)$ such that*

- (a) $\partial(p; v) = P_v(p; 1)$;
- (b) $\partial(A \cdot B) = (\partial A) \cdot B + (-1)^k A \cdot \partial B$ for $A, B \in \mathcal{A}_\bullet^s(p)$ where $\dim(A) = k$;
- (c) $\partial \circ \partial = 0$.

The operator ∂ additionally satisfies

- (d) $\partial(p; \alpha) = \sum_i P_{e_i} E_{e_i}^\dagger(p; \alpha)$, where $\{e_i\}$ forms an orthonormal basis of \mathbb{R}^n .

Proof. The first part is virtually identical to the proof of existence of exterior derivative and we omit it. Part (d) is easy to prove using the basis $\{e_I\}$ for $\Lambda_k(\mathbb{R}^n)$ and properties (a)-(c). \square

¹¹a formal treatment of support will be given in §5.

Let $\mathcal{A}_k^s(\mathbb{R}^n)$ be the free space $\mathbb{R}\langle \cup_p \mathcal{A}_k^s(p) \rangle$. In particular, an element of $\mathcal{A}_k^s(\mathbb{R}^n)$ is a formal sum $\sum_{i=1}^m (p_i; \sigma^i \otimes \alpha_i)$ where $p_i \in \mathbb{R}^n$ (see 3.4.2). It follows from Proposition 3.4.1 that $\mathcal{A}_k^s(\mathbb{R}^n)$ is isomorphic to the free space $(S^s \otimes \Lambda_k) \langle \mathbb{R}^n \rangle$. Now $\mathcal{A}_k^s(\mathbb{R}^n)$ is naturally included in $\hat{\mathcal{B}}_k^{s+1}(\mathbb{R}^n)$, and is thus endowed with the subspace topology¹².

Definition 3.4.6. Let $Ch_k^s(\mathbb{R}^n) := \overline{(\mathcal{A}_k^s(\mathbb{R}^n), B^{s+1})}$. Elements of $Ch_k^s(\mathbb{R}^n)$ are called *k-chainlets*¹³ of order *s*.

We will see in the next section that $\{Ch_k^s(\mathbb{R}^n) \subset \hat{\mathcal{B}}_k^{s+1}(\mathbb{R}^n)\}$ forms a bigraded chain complex.

3.5 Boundary and the generalized Stokes' Theorem

Theorem 3.5.1. For each $r \geq 0$ and $0 \leq k \leq n$, there exists a unique continuous bigraded operator $\partial : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_{k-1}^{r+1}(\mathbb{R}^n)$ such that

- (a) $\partial \sum (p_i; v_i) = P_{v_i}(p_i; 1);$
- (b) $\partial((p; \alpha) \cdot (p; \beta)) = (\partial(p; \alpha)) \cdot (p; \beta) + (-1)^k (p; \alpha) \cdot \partial(p; \beta)$ for $(p; \alpha), (p; \beta) \in \mathcal{A}(p)$ with $\dim(p; \alpha) = k;$
- (c) $\partial \circ \partial = 0.$

This operator ∂ additionally satisfies

- (d) $\partial(\mathcal{A}_k^s(\mathbb{R}^n)) \subset \mathcal{A}_{k-1}^{s+1}(\mathbb{R}^n)$ for all $k \geq 1$ and $s \geq 0;$
- (e) $\partial(\mathcal{A}_0^s(\mathbb{R}^n)) = \{0\};$
- (f) $\partial = \sum_i P_{e_i} E_{e_i}^\dagger.$

Proof. Parts (a)-(f) follow directly from Lemma 3.4.5, while continuity is a consequence of (f) and continuity of P_v and E_v^\dagger (see Theorems 3.2.2 and 3.3.5). \square

Theorem 3.5.2. The linear map $\partial : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_{k-1}^{r+1}(\mathbb{R}^n)$ is continuous with $\|\partial J\|_{B^{r+1}} \leq kn \|J\|_{B^r}$ for all $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$. It therefore extends to a continuous operator $\partial : \hat{\mathcal{B}}(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}(\mathbb{R}^n)$ and restricts to the chainlet complex $Ch_k^s(\mathbb{R}^n)$.

Theorem 3.5.3. The following relations hold:

- (a) $\{\partial, E_v\} = P_v$ (Cartan's magic formula for differential chains);
- (b) $[E_v^\dagger, \partial] = 0;$
- (c) $[P_v, \partial] = 0.$

¹²The inner product we chose on \mathbb{R}^n induces an inner product on the exterior algebra $\langle \cdot, \cdot \rangle_\wedge$ using determinant (see Definition 2.1.1). It induces an inner product on the symmetric algebra $\langle \cdot, \cdot \rangle_\circ$ using permanent of a matrix $\langle \sigma, \tau \rangle_\circ := \text{per}(\langle u_r, v_s \rangle)$, and thus on $\mathcal{A}_k^s(p)$ via $\langle p; \sigma \otimes \alpha, p; \tau \otimes \beta \rangle_\otimes := \langle \sigma, \tau \rangle_\circ \langle \alpha, \beta \rangle_\wedge$. However, this inner product $\langle \cdot, \cdot \rangle_\otimes$ on $\mathcal{A}_k^s(p)$ does not extend to a continuous inner product on $\mathcal{A}_k^s(\mathbb{R}^n)$, although it can be useful for computations on Dirac chains of arbitrary order and dimension as long as limits are not taken.

¹³Differential chains of class \mathcal{B} were originally called chainlets. It is only recently that the author has begun to appreciate the importance of what we now call the chainlet complex, which is a subcomplex of the differential chain complex.

Proof. This follows from Proposition 3.2.3. □

Theorem 3.5.4 (Stokes' Theorem). *Let $0 \leq k \leq n-1$. Then*

$$\int_{\partial J} \omega = \int_J d\omega$$

for all matching pairs $J \in \hat{\mathcal{B}}_{k+1}^{r-1}(\mathbb{R}^n)$, $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$, and $1 \leq r \leq \infty$.

Proof. By Lemma 3.2.4 and Corollary 3.3.8 $\omega \partial(J) = \omega \sum_{i=1}^n P_{e_i} E_{e_i}^\dagger(J) = \sum_{i=1}^n L_{e_i} \omega E_{e_i}^\dagger(J) = \sum_{i=1}^n de_i \wedge L_{e_i} \omega(J)$. Since $d\omega = \sum_{i=1}^n de_i \wedge L_{e_i} \omega$ is equivalent to the classical definition of $d\omega$, the result follows. □

Lemma 3.5.5. *Let W be a bounded open subset of \mathbb{R}^n with a piecewise smooth frontier. Then $\partial \widetilde{W} = \widetilde{\partial W}$.*

Proof. Since W is bounded, it is represented by $\widetilde{W} \in \hat{\mathcal{B}}_n^1(\mathbb{R}^n)$. By the classical Stokes' theorem for open sets with piecewise smooth boundaries, Stokes' theorem for differential chains 3.5.4 and Theorem 2.9.4,

$$\int_{\partial \widetilde{W}} \omega = \int_{\widetilde{W}} d\omega = \int_W d\omega = \int_{\partial W} \omega = \int_{\widetilde{\partial W}} \omega$$

for all $\omega \in \mathcal{B}_{n-1}^2(\mathbb{R}^n)$. Hence $\partial \widetilde{W} = \widetilde{\partial W}$. □

Example 3.5.6. Algebraic boundary of the Cantor set. *In Example 2.10.7 we found a differential 1-chain $\Gamma = \lim_{n \rightarrow \infty} (3/2)^n \widetilde{E}_n$ representing the middle third Cantor set where $E_n = \cup [qp_{n,i}, q_{n,i}]$ is the set obtained after removing middle thirds at the n -th stage. Then $\partial \Gamma = \sum_{n=1}^{\infty} \sum_i (q_{n,i}; (3/2)^n) - (p_{n,i}; (3/2)^n)$. Furthermore, $\int_{\partial \Gamma} \omega = \int_{\Gamma} d\omega$ for all $\omega \in C^{1+Lip}$. For example,*

$$\int_{\partial \Gamma} x = \int_{\Gamma} dx = \lim_{n \rightarrow \infty} \int_{(3/2)^n \widetilde{E}_n} dx = 1.$$

3.5.1 Directional boundary and directional exterior derivative

Definition 3.5.7. *Let $v \in \mathbb{R}^n$. Define the **directional boundary** on differential chains by $\partial_v : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_{k-1}^{r+1}(\mathbb{R}^n)$ by $\partial_v := P_v E_v^\dagger$. See Figure 4.*

For example, $\partial_{e_1}(p; 1 \otimes e_1) = (p; e_1 \otimes 1)$, $\partial_{e_1}(p; 1 \otimes e_1 \wedge e_2) = (p; e_1 \otimes e_2)$ and $\partial_{e_2}(p; 1 \otimes e_1 \wedge e_2) = (p; -e_2 \otimes e_1)$.

Definition 3.5.8. *For $v \in \mathbb{R}^n$, define the **directional exterior derivative** on differential forms $d_v \omega = v^\flat \wedge L_v \omega$.*

Theorem 3.5.9. *$d_v : \mathcal{B}_{k-1}^{r+1}(\mathbb{R}^n) \rightarrow \mathcal{B}_k^r(\mathbb{R}^n)$ satisfies $d_v \omega = \omega \partial_v$.*

Proof. Since $v^\flat \wedge L_v \omega = L_v \omega E_v^\dagger$, we have $d_v \omega = v^\flat \wedge L_v \omega = \omega P_v E_v^\dagger$ by Lemma 3.3.7. □

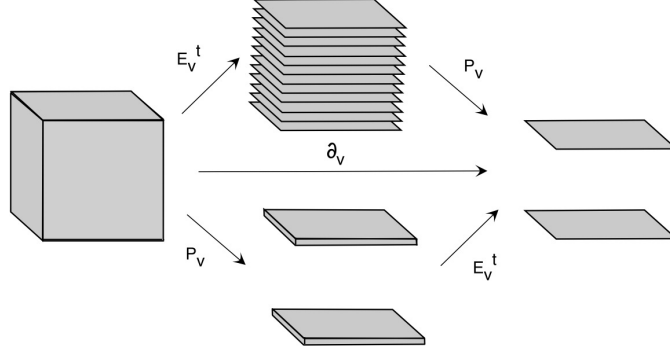


Figure 4: Directional boundary ∂_v

3.6 Perpendicular complement and classical integral theorems of calculus

3.6.1 Clifford algebra and perpendicular complement

Consider the subalgebra¹⁴ $\mathcal{Cl} \subset \mathcal{L}(\hat{\mathcal{B}})$ of linear operators on $\hat{\mathcal{B}}(\mathbb{R}^n)$ generated by $\{C_v : v \in \mathbb{R}^n\}$ where $C_v = E_v + E_v^\dagger$. According to the bilinearity of E and E^\dagger (see Corollaries 3.1.5 and 3.2.5), we know that \mathcal{Cl} is isomorphic to the Clifford algebra. However, the isomorphism depends on the inner product. The algebra product¹⁵ is composition of operators: $C_u \cdot C_v = (E_u + E_u^\dagger) \circ (E_v + E_v^\dagger)$. It is worth understanding what C_v does to a simple k -element $(p; \alpha)$: For simplicity, assume $\alpha = e_I$ and $v = e_i$, taken from an orthonormal basis $\{e_i\}$ of \mathbb{R}^n . If e_i is in the k -direction of e_I , then C_{e_i} “divides it” out of e_I , reducing its dimension to $k - 1$. If e_i is not in the k -direction of e_I , then C_{e_i} “wedges it” to e_I , increasing its dimension to $k + 1$.

Definition 3.6.1. Let $\perp : \mathcal{A}_k(\mathbb{R}^n) \rightarrow \mathcal{A}_{n-k}(\mathbb{R}^n)$ be the operator on Dirac chains given by $\perp := C_{e_n} \circ \cdots \circ C_{e_1} = \prod_{i=1}^n (E_{e_i} + E_{e_i}^\dagger)$. Then \perp extends to a continuous linear map on $\hat{\mathcal{B}}(\mathbb{R}^n)$ since E_v and E_v^\dagger are continuous. We call \perp **perpendicular complement**. Perpendicular complement does depend on the inner product, but not on the choice of orthonormal basis.

It is not too hard to see that \perp behaves as we expect. That is,

Proposition 3.6.2. If α is a k -vector, then $\perp(p; \alpha) = (p; \alpha^\perp)$, where α^\perp is the $(n - k)$ -vector satisfying $\alpha \wedge \alpha^\perp = (-1)^k \|\alpha\|^2 e_1 \wedge \cdots \wedge e_n$. Furthermore, the k -direction of α is orthogonal to the $(n - k)$ -direction of α^\perp . Thus, $\perp \circ \perp = (-1)^{k(n-k)} I$.

It follows that $\star \omega := \omega \perp$ where \star is the classical Hodge [Hod89] star operator on forms.

Theorem 3.6.3 (Star theorem). Let $0 \leq k \leq n$. The linear map $\perp : \mathcal{A}_k(\mathbb{R}^n) \rightarrow \mathcal{A}_{n-k}(\mathbb{R}^n)$ determined by $(p; \alpha) \mapsto (p; \perp \alpha)$, for simple k -elements $(p; \alpha)$, satisfies

$$\|\perp A\|_{B^r} = \|A\|_{B^r}$$

¹⁴The author thanks Patrick Barrow for pointing out that this operator algebra is isomorphic to the Clifford algebra.

¹⁵Some authors call $u \wedge v + \langle u, v \rangle$ the “geometric product”. This is found naturally as $(E_u + E_u^\dagger) \circ (E_v + E_v^\dagger)(0; 1) = u \wedge v + \langle u, v \rangle$.

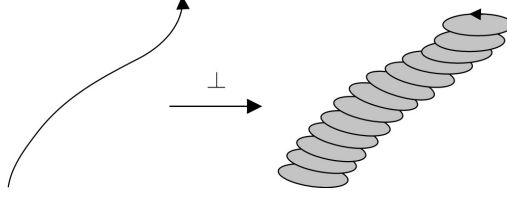


Figure 5: Perpendicular complement \perp

for all $A \in \mathcal{A}_k(\mathbb{R}^n)$. It therefore extends to a continuous linear map $\perp: \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_{n-k}^r(\mathbb{R}^n)$ for each $0 \leq k \leq n, 0 \leq r \leq \infty$, and to a continuous graded operator $\perp \in \mathcal{L}(\hat{\mathcal{B}})$ satisfying

$$\int_{\perp J} \omega = \int_J \star \omega$$

for all matching pairs $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, $\omega \in \mathcal{B}_{n-k}^r(\mathbb{R}^n)$, and $0 \leq r \leq \infty$.

Proof. We know $\|\perp J\|_{B^r} = \|J\|_{B^r}$ since $\|J\|_{B^r} = \sup \frac{|\omega(J)|}{\|\omega\|_{B^r}}$ and $\|\omega\|_{B^r} = \|\star \omega\|_{B^r}$ for $0 \leq r < \infty$. A direct proof using difference chains is straightforward and can be found in [Har06].

The integral relation holds on Dirac chains since $\star \omega = \omega \perp$ on Dirac chains and by continuity. Continuity of $\perp: \hat{\mathcal{B}}(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}(\mathbb{R}^n)$ follows. □

3.6.2 Geometric coboundary, Laplace, and Dirac operators

Definition 3.6.4. Define the **coboundary operator** $\diamond := \perp \partial \perp$. Since ∂ decreases dimension, \diamond increases dimension. Its dual operator is the codifferential δ where $\delta = \star d \star$.

Definition 3.6.5. Define the **geometric¹⁶ Laplace operator** $\square := \diamond \partial + \partial \diamond$.

This operator preserves dimension and restricts to an operator on $\hat{\mathcal{B}}_k(\mathbb{R}^n)$. The dual operator is the classical Laplace operator $\Delta = d\delta + \delta d$ on differential forms. The *geometric Dirac operator* $\partial + \diamond$ dualizes to the Dirac operator $d + \delta$ on forms. (see [Har06]).

Theorem 3.6.6 (Gauss-Green Divergence Theorem). *Let $1 \leq k \leq n$. The integral relation*

$$\int_J d \star \omega = \int_{\perp \partial J} \omega$$

holds for all matching pairs $J \in \hat{\mathcal{B}}_k^{r-1}(\mathbb{R}^n)$, $\omega \in \mathcal{B}_{n-k+1}^r(\mathbb{R}^n)$, and $1 \leq r \leq \infty$.

Proof. This follows directly from Theorems 3.5.4 and 3.6.3. □

Example 3.6.7. If \tilde{S} represents an oriented surface S with boundary in \mathbb{R}^3 and ω is a smooth 2-form defined on S , Theorem 3.6.6 tells us $\int_{\tilde{S}} d \star \omega = \int_{\perp \partial \tilde{S}} \omega$. This result equates the net divergence of ω within S with the net flux of ω (thought of as a 2-vector field with respect to the inner product) across the boundary of S .

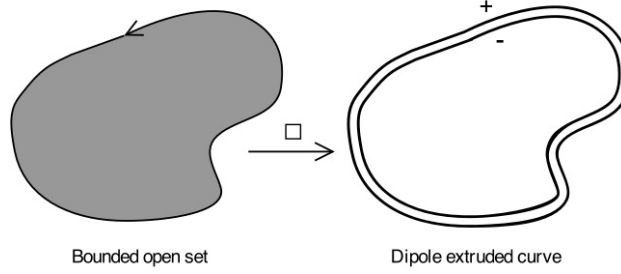


Figure 6: Geometric Laplace operator \square of an open set in \mathbb{R}^2

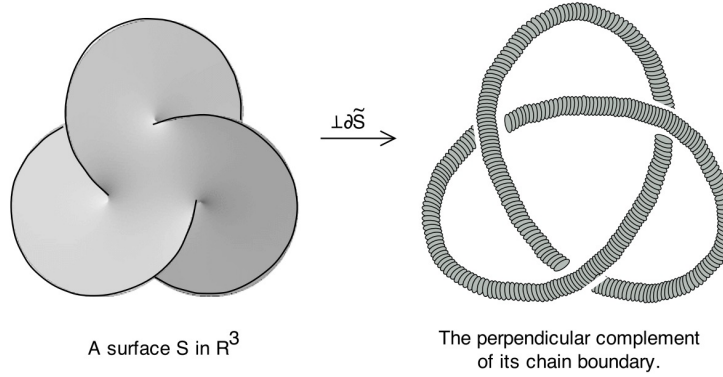


Figure 7: A domain of the divergence theorem for surfaces in \mathbb{R}^3

Corollary 3.6.8 (Kelvin-Stokes' Curl Theorem). *Let $1 \leq k \leq n$. The integral relation*

$$\int_{\partial J} \omega = \int_{\perp J} \star d\omega$$

holds for all matching pairs $J \in \hat{\mathcal{B}}_k^{r-1}(\mathbb{R}^n)$, $\omega \in \mathcal{B}_{k-1}^r(\mathbb{R}^n)$, and $1 \leq r \leq \infty$.

Proof. This follows directly from Theorems 3.5.4 and 3.6.3. □

Corollary 3.6.9 (Higher order divergence theorem). *Let $s \geq 0$ and $0 \leq k \leq n$. The integral relation*

$$\int_{\square^s J} \omega = \int_J \Delta^s \omega$$

holds for all matching pairs $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, $\omega \in \mathcal{B}_k^{r+2s}(\mathbb{R}^n)$, and $0 \leq r \leq \infty$.

Figure 6 shows this Corollary 3.6.9 has deep geometric meaning for $s = 1$. Whereas the divergence theorem equates net flux of a k -vector field across an oriented boundary with net interior divergence, Corollary 3.6.9 equates net

¹⁶Yes, this is an abuse of the word “geometric.” The author is amenable to suggestions about what to call this operator.

flux of the “orthogonal differential” of a k -vector field across an oriented boundary with net interior “second order divergence”. In this sense Corollary 3.6.9 is a “higher order divergence theorem”. As far as we know, it has no classical precedent.

The following diagram depicts the underpinnings of the *exterior differential complex*. The “boundary” operators are boundary ∂ (boundary, predual to exterior derivative), \diamond (geometric coboundary, predual to $*d*$), extrusion E_v , and retraction E_v^\dagger . The commutation relations are given in Propositions 3.1.3, 3.2.3, 3.3.6, 3.5.3, and 3.6.2.

4 Multiplication by a function and partitions of unity

4.1 Multiplication by a function and change of density

We prove that the topological vector space of differential chains $\hat{\mathcal{B}}(\mathbb{R}^n)$ is a graded module over the ring of functions (0-forms) $\mathcal{B}_0(\mathbb{R}^n)$.

Lemma 4.1.1. *The Banach space $\mathcal{B}_0^r(\mathbb{R}^n)$ is a ring with unity via pointwise multiplication of functions. Specifically, if $f, g \in \mathcal{B}_0^r(\mathbb{R}^n)$, then $\|f \cdot g\|_{B^r} \leq n2^r \|f\|_{B^r} \|g\|_{B^r}$.*

Proof. The unit function $u(x) = 1$ is an element of $\mathcal{B}_0^r(\mathbb{R}^n)$ for $r \geq 0$ since $\|u\|_{B^r} = 1$. The proof $\|f \cdot g\|_{B^r} \leq 2^r \|f\|_{B^r} \|g\|_{B^r}$ for $n = 1$ is a straightforward application of the product rule and we omit it. The general result follows by taking coordinates. \square

Definition 4.1.2. *Define the bilinear map*

$$\begin{aligned} m : \mathcal{B}_0^r(\mathbb{R}^n) \times \mathcal{A}_k(\mathbb{R}^n) &\rightarrow \mathcal{A}_k(\mathbb{R}^n) \\ (f, (p; \alpha)) &\mapsto (p; f(p)\alpha) \end{aligned}$$

which we call **multiplication by a function**. Let $m_f(p; \alpha) := m(f, (p; \alpha))$.

Lemma 4.1.3. *If $v \in \mathbb{R}^n$ and $f \in \mathcal{B}_0^r(\mathbb{R}^n)$, $r \geq 1$, then $E_{fv} = m_f E_v$.*

Proof. This follows directly from the definitions:

$$E_{fv}(p; \alpha) = (p; f(p)v \wedge \alpha) = m_f(p; v \wedge \alpha) = m_f E_v(p; \alpha).$$

□

Theorem 4.1.4. *Let $0 \leq k \leq n$ and $r \geq 0$. The bilinear map $m = m_k^r : \mathcal{B}_0^r(\mathbb{R}^n) \times \mathcal{A}_k(\mathbb{R}^n) \rightarrow \mathcal{A}_k(\mathbb{R}^n)$ extends to a separately continuous bilinear map $m = m_k^r : \mathcal{B}_0^r(\mathbb{R}^n) \times \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ with*

$$\|m(f, J)\|_{B^r} \leq n2^r \|f\|_{B^r} \|J\|_{B^r} \text{ for all } J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n).$$

Furthermore, for each $0 \leq k \leq n$, there exists a separately continuous bilinear map $m = m_k : \mathcal{B}_0(\mathbb{R}^n) \times \hat{\mathcal{B}}_k(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k(\mathbb{R}^n)$ which restricts to m_k^r on each $\mathcal{B}_0^r(\mathbb{R}^n) \times u_k^r(\hat{\mathcal{B}}_k^r(\mathbb{R}^n))$.

Proof. We first prove the inequality for nonzero $A \in \mathcal{A}_k(\mathbb{R}^n)$. By Lemma 4.1.1 $\frac{|f_{m_f A} \omega|}{\|\omega\|_{B^r}} \leq \frac{\|f \cdot \omega\|_{B^r} \|A\|_{B^r}}{\|\omega\|_{B^r}} \leq nr \|f\|_{B^r} \|A\|_{B^r}$. The inequality follows since $\|m_f A\|_{B^r} = \sup \frac{|f_{m_f A} \omega|}{\|\omega\|_{B^r}}$. We can therefore extend m_k^r to $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ via completion, and the operator m_k^r will still satisfy the inequality.

For the last assertion, first fix $f \in \mathcal{B}_0(\mathbb{R}^n)$. Since $m_f(H_k(\mathbb{R}^n) \subset H_k(\mathbb{R}^n))$, we can apply Theorem 2.11.14 to extend $m_f : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ to a continuous linear map $m_f : \hat{\mathcal{B}}_k(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k(\mathbb{R}^n)$. Define $m : \mathcal{B}_0(\mathbb{R}^n) \times \hat{\mathcal{B}}_k(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k(\mathbb{R}^n)$ by $m(f, J) := m_f(J)$. This is well-defined since $f \in \mathcal{B}_0(\mathbb{R}^n)$ implies $f \in \mathcal{B}_0^r(\mathbb{R}^n)$.

Last of all, we establish continuity in the first variable: Suppose $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$. Then there exists $r \geq 0$ and $J^r \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ such that $u_k^r(J^r) = J$. Suppose $f_i \rightarrow 0$ in $\mathcal{B}_0(\mathbb{R}^n)$. Since the inclusion $\mathcal{B}_0(\mathbb{R}^n) \rightarrow \mathcal{B}_0^r(\mathbb{R}^n)$ is the identity map and continuous, then $f_i \rightarrow 0$ in $\mathcal{B}_0^r(\mathbb{R}^n)$. Now use the fact that $m_k^r : \mathcal{B}_0^r(\mathbb{R}^n) \times \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ is continuous in the first variable. □

Proposition 4.1.5. *Let $f, g \in \mathcal{B}_0^r(\mathbb{R}^n)$, $0 \leq r \leq \infty$, and $v \in \mathbb{R}^n$. Then*

$$(a) [m_f, m_g] = [m_f, E_v] = [m_f, E_v^\dagger] = 0;$$

$$(b) [m_f, P_v] = m_{L_v f};$$

$$(c) [m_f, \partial] = \sum_i df(e_i) E_{e_i}^\dagger.$$

Proof. (a): These follow directly from the definitions. (b): By the Mean Value Theorem there exists $q_t = p + stv$, $0 \leq s \leq 1$ such that $\frac{f(p+tv) - f(p)}{t} = L_v f(q_t)$. Then

$$\begin{aligned} m_f P_v(p; \alpha) &= m_f \lim_{t \rightarrow 0} (p + tv; \alpha/t) - (p; \alpha/t) = \lim_{t \rightarrow 0} (p + tv; f(p + tv)\alpha/t) - (p; f(p)\alpha/t) \\ &= \lim_{t \rightarrow 0} (p + tv; (f(p) + tL_v f(q_t))\alpha/t) - (p; f(p)\alpha/t) \\ &= \lim_{t \rightarrow 0} (p + tv; f(p)\alpha/t) - (p; f(p)\alpha/t) + \lim_{t \rightarrow 0} (p; L_v f(q_t)\alpha) \\ &= P_v m_f(p; \alpha) + m_{L_v f}(p; \alpha). \end{aligned}$$

(c): This uses (a) and (b), as well as Theorem 3.5.1 (f).

$$\begin{aligned}
m_f \partial - \partial m_f &= m_f \sum_i P_{e_i} E_{e_i}^\dagger - \sum_i P_{e_i} E_{e_i}^\dagger m_f \\
&= m_f \sum_i P_{e_i} E_{e_i}^\dagger - \sum_i P_{e_i} m_f E_{e_i}^\dagger \\
&= \left(m_f \sum_i P_{e_i} - \sum_i P_{e_i} m_f \right) E_{e_i}^\dagger \\
&= \left(\sum_i m_f P_{e_i} - P_{e_i} m_f \right) E_{e_i}^\dagger \\
&= \left(\sum_i m_{L_{e_i} f} \right) E_{e_i}^\dagger
\end{aligned}$$

By Cartan's magic formula $L_{e_i} = i_{e_i} d + di_{e_i}$. Hence $L_{e_i} f = i_{e_i} df + di_{e_i} f = i_{e_i} df = df(e_i)$. The result follows. \square

We denote the (continuous) dual operator by $f \cdot \in \mathcal{L}(\mathcal{B}(\mathbb{R}^n))$ where $f \in \mathcal{B}_0(\mathbb{R}^n)$.

Theorem 4.1.6 (Change of density). *Let $0 \leq k \leq n$. The space of differential chains $\hat{\mathcal{B}}_k(\mathbb{R}^n)$ is a graded module over the ring $\mathcal{B}_0(\mathbb{R}^n)$ satisfying*

$$\int_{m_f J} \omega = \int_J f \cdot \omega \quad (4.1)$$

for all matching pairs $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$, and $0 \leq r \leq \infty$.

Proof. Since $\omega(m_f(p; \alpha)) = f \cdot \omega(p; \alpha)$ the integral relation holds for Dirac chains, and thus to pairs of all chains and forms of matching class by separate continuity of the integral pairing of Theorem 4.1.4 (for $0 \leq r < \infty$) and Corollary 2.8.4 (for $r = \infty$). \square

We can actually say a bit more about the continuity of m_f with respect to $f \in \mathcal{B}_0^r(\mathbb{R}^n)$. Convergence in $\mathcal{B}_0^r(\mathbb{R}^n)$ is restrictive as it does not lend itself nicely to bump functions and partitions of unity. In particular, if $\{\phi_i\}$ is a partition of unity, then $\sum_{i=1}^N \phi_i \rightarrow 1$ in the B^r norm¹⁷. In §5 we need m_f to be continuous under a more local notion of convergence (the B^r version of the compact-open topology.) The following lemma and its proof are due to Harrison Pugh.

Lemma 4.1.7. *Let $f_i, f \in \mathcal{B}_0^r(\mathbb{R}^n)$, $r \geq 0$, such that $f_i \rightarrow f$ pointwise and $\|f - f_i\|_{B^r} < C_r$ for some C_r independent of i . Then $m_{f_i} J \rightarrow m_f J$ in the B^r -norm for all $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$. Suppose $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$. If $f_i, f \in \mathcal{B}_0(\mathbb{R}^n)$, $f_i \rightarrow f$ pointwise and there exists C_r with $\|f - f_i\|_{B^r} < C_r$ for all $r \geq 0$, then $m_{f_i} J \rightarrow m_f J$ in $\hat{\mathcal{B}}_k(\mathbb{R}^n)$.*

Proof. Let $\epsilon > 0$. We show there exists N such that if $i > N$ then $\|m_f J - m_{f_i} J\|_{B^r} < \epsilon$. Suppose not. Then for all N there exists $i_N > N$ and $\omega_N \in \mathcal{B}_k^r(\mathbb{R}^n)$ with $\|\omega_N\|_{B^r} = 1$ such that

$$((f - f_{i_N})\omega_N)(J) \geq \epsilon.$$

¹⁷The author is grateful to H. Pugh for pointing this out.

(This is by the definition of the B^r norm on chains as a supremum over forms of norm 1.) Now, let $A_i \rightarrow J$ be Dirac chains. Then there exists M such that if $j > M$, then $\omega(J) - \omega(A_i) < \epsilon/2$ for all $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$ with $\|\omega\|_{B^r} < nrC$. Therefore, putting these two together, and using the fact that $\|((f - f_{i_N})\omega_N)\|_{B^r} \leq nr\|f - f_i\|_{B^r}\|\omega_N\|_{B^r} < nrC$, we have

$$((f - f_{i_N})\omega_N)(A_j) > \epsilon/2$$

for all N . Now, fix such a j . Say $A_j = \sum_s (p_s; \alpha_s)$. Then

$$((f - f_{i_N})\omega_N)(A_j) = \sum_s (f(p_s) - f_{i_N}(p_s))\omega_N(p_s; \alpha_s).$$

Since $f_i \rightarrow f$ pointwise, we can make N large enough such that

$$|f(p_s) - f_{i_N}(p_s)| < \frac{\epsilon}{2 \max \|\alpha_s\|_0}$$

for all s . Therefore,

$$\sum_s (f(p_s) - f_{i_N}(p_s))\omega_N(p_s; \alpha_s) < \epsilon/2,$$

which is a contradiction. The last part follows by applying the first to $J^r \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ with $u_k^r(J^r) = J$. Then $m_{f_i}J \rightarrow m_fJ$ in the B^s norm for each $s \geq r$. \square

4.2 Partitions of unity

If $A \in \mathcal{A}_k(\mathbb{R}^n)$ is a Dirac chain, and $W \subseteq \mathbb{R}^n$ is open, let $A|_W$ be the *restriction of A to W* . That is, if $A = \sum_{i=1}^s (p_i; \alpha_i)$, then $A|_W = \sum_j (p_{i_j}; \alpha_{i_j})$ where the sum is taken over all $p_{i_j} \in W$.

Example 4.2.1. Let Q be the open unit disk in \mathbb{R}^2 . For each integer $m \geq 1$, let $A_m = ((1 + 1/2m, 0); m) - ((1 - 1/2m, 0); m)$ and $B_m = A_m - ((1, 0); e_1 \otimes 1)$. Then $B_m \rightarrow 0$ as $m \rightarrow \infty$ in $\hat{\mathcal{B}}_0^2(\mathbb{R}^2)$, and yet $B_m|_Q = (1 - 1/2m; -m)$ diverges as $m \rightarrow \infty$. This example shows that it can be problematic to define the part of a chain in every open set. We will see however in (see §6.3) that “inclusion” is well-defined, so it is more natural to use cosheaves with differential chains, as opposed to sheaves.

Theorem 4.2.2. Suppose $\{U_i\}_{i=1}^\infty$ is a locally finite bounded open cover of \mathbb{R}^n and $\{\Phi_i\}_{i=1}^\infty$ is a partition of unity subordinate to $\{U_i\}_{i=1}^\infty$ with $\Phi_i \in \mathcal{B}_0^r(\mathbb{R}^n)$. If $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, then

$$J = \sum_{i=1}^\infty (m_{\Phi_i} J)$$

where the convergence is in the B^r norm. If $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$, then

$$J = \sum_{i=1}^\infty (m_{\Phi_i} J)$$

where the convergence is in the inductive limit topology.

Proof. Apply Pugh's Lemma¹⁸ 4.1.7 to $\sum_{i=1}^N \Phi_i$ and the function $1 \in \mathcal{B}_0(\mathbb{R}^n)$. Then $\sum_{i=1}^N (m_{\Phi_i} J) = (\sum_{i=1}^N m_{\Phi_i}) J \rightarrow J$ in the B^r norm, assuming $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, or the inductive limit topology, assuming $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$. \square

We immediately deduce

Corollary 4.2.3 (Fundamental Lemma). *If $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ satisfies $m_f J = 0$ for all $f \in \mathcal{B}_0(\mathbb{R}^n)$ with compact support, then $J = 0$.*

This implies the fundamental lemma in the calculus of variations. For $0 = \int_U f \cdot g dV = \int_{m_f \tilde{U}} g dV$ for all g implies $m_f \tilde{U} = 0$.

4.2.1 Partitions of unity and integration

If $I = (0, 1)$ and $\{\phi_s\}$ is a partition of unity subordinate to a covering of $I = \cup_{s=1}^k I_s$, then we can write the chain representative \tilde{I} of I as a sum $\tilde{I} = \sum_{s=1}^k m_{\phi_s} \tilde{I}_s$. This ties the idea of an atlas and its overlap maps to integration. Roughly speaking, the overlapping subintervals have varying density, but add up perfectly to obtain the unit interval. We obtain $\int_{\tilde{I}} \omega = \sum_{s=1}^k \int_{m_{\phi_s} \tilde{I}_s} \omega$. This is in contrast with our discrete method of approximating \tilde{I} with Dirac chains A_m to calculate the same integral $\int_{\tilde{I}} \omega = \lim \int_{A_m} \omega$.

5 Support

We have already seen the *support* $\text{supp}(\omega)$ of a differential form $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$ in Definition (2.4.1) and the *support* $\text{supp}(A)$ of a Dirac chain $A \in \mathcal{A}_k(\mathbb{R}^n)$ in §(1.1). We show that associated to each nonzero chain $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$ is a well-defined nonempty closed subset $\text{supp}(J) \subseteq \mathbb{R}^n$ called the *support* of J .

Definition 5.0.4. Let $\Omega_\epsilon(p) := \{q \in \mathbb{R}^n : \|p - q\| < \epsilon\}$, the open ball of radius $\epsilon > 0$ about p .

Definition 5.0.5. If $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, $0 \leq r \leq \infty$, let

$$\text{supp}(J) := \left\{ p \in \mathbb{R}^n : \text{for all } \epsilon > 0, \text{ there exists } \eta \in \mathcal{B}_k^r(\mathbb{R}^n) \text{ s.t. } \int_J \eta \neq 0 \text{ where } \text{supp}(\eta) \subset \Omega_\epsilon(p) \right\}.$$

Proposition 5.0.6. Let $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$. Then $\text{supp}(J) = \text{supp}(u_k^{r,s}(J)) = \text{supp}(u_k^r(J))$ for all $0 \leq r \leq s$.

Proof. Suppose $p \in \text{supp}(J)$ and $\epsilon > 0$. Then there exists $\eta \in \mathcal{B}_k^r(\mathbb{R}^n)$ with $\text{supp}(\eta) \subset \Omega_\epsilon(p)$ and $\int_J \eta \neq 0$. By Theorem 2.11.2 η can be approximated by $\xi \in \mathcal{B}_k(\mathbb{R}^n)$ with $\text{supp}(\xi) \subset \Omega_\epsilon(p)$ and $\int_J \xi \neq 0$. Thus $p \in \text{supp}(u_k^r(J))$, as well as $\text{supp}(u_k^{r,s}(J))$. \square

By working with forms supported in ϵ -neighborhoods of points, it is not hard to see that the support of a chain agrees with our previous definition of the support of a Dirac chain A .

Support is a map from $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ to the power set of \mathbb{R}^n . It is not linear, nor is it continuous in the Hausdorff metric. Consider, for example, $A_t = (p; t\alpha) + (q; (1-t)\alpha)$. Then $\text{supp}(A_t) = \{p, q\}$ for all $0 < t < 1$, but $\text{supp}(A_0) = q$ and $\text{supp}(A_1) = p$. However, we can say something about the support of limits of differential chains in Theorem 5.0.13

¹⁸This is a much more elegant approach than the author's original proof using weak convergence in \mathcal{D}' .

If $J = 0$, then $\text{supp}(J) = \emptyset$. If X is any subset of \mathbb{R}^n with $\text{supp}(J) \subseteq X$, then we say J is *supported in X* . If $\text{supp}(J)$ is compact, we say that J has *compact support*.

Theorem 5.0.7. *If J is a nonzero differential chain, then $\text{supp}(J)$ is a closed, nonempty set.*

Proof. Suppose $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ and $\text{supp}(J) = \emptyset$. Then for each $p \in \mathbb{R}^n$ there exists $\epsilon > 0$ such that $f_J \omega = 0$ for all $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$ with $\text{supp}(\omega) \subset \Omega_\epsilon(p)$. We can choose a locally finite subcover of \mathbb{R}^n by such open balls $\Omega_\epsilon(p)$. Let $\{f_i\}$ be a partition of unity subordinate to this cover. Then $\sum_i (m_{f_i} J)$ converges to J in the B^r norm by Theorem 4.1.4. Hence $f_J \omega = \sum_i f_{m_{f_i} J} \omega = \sum_i f_J f_i \cdot \omega = 0$ for all $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$. Thus $J = 0$. Now suppose $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$ and $\text{supp}(J) = \emptyset$. Then there exists $J^r \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ with $u_k^r(J^r) = J$. By Proposition 5.0.6 we know $\text{supp}(J) = \text{supp}(J^r) = \emptyset$. By the first part, $J^r = 0$ and thus $J = u_k^r(J^r) = 0$. Also say that we use $\hat{\mathcal{B}}$ instead of $\hat{\mathcal{B}}^\infty$ to keep us mindful that our topology is not the Banach space which is obtained by taking limits of norms. It also reminds us that the space is predual to \mathcal{B} .

We show $\text{supp}(J)$ is closed: Suppose $p_i \rightarrow p$ and $p_i \in \text{supp}(J)$. Let $\epsilon > 0$. Then $d(p_i, p) < \epsilon/2$ for sufficiently large i . Since $p_i \in \text{supp}(J)$, there exists $\eta \in \mathcal{B}_k^r(\mathbb{R}^n)$, respectively, $\eta \in \mathcal{B}_k(\mathbb{R}^n)$, supported in $\Omega_{\epsilon/2}(p)$ such that $f_J \eta \neq 0$. Thus $p \in \text{supp}(J)$ since $\Omega_{\epsilon/2}(p) \subset \Omega_\epsilon(p)$. \square

Lemma 5.0.8. *Suppose the differential chain J and the differential form η are a matching pair with $\text{supp}(\eta) \subset \text{supp}(J)^c$. Then $f_J \eta = 0$.*

Proof. Suppose $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, $\eta \in \mathcal{B}_k^r(\mathbb{R}^n)$, $0 \leq r < \infty$, and $\text{supp}(\eta) \subset \text{supp}(J)^c$. For each $p \in \text{supp}(\eta)$, there exists $\epsilon > 0$ with $\Omega_\epsilon(p) \cap \text{supp}(J) = \emptyset$. Thus $f_J \omega = 0$ for all $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$ with $\text{supp}(\omega) \subset \Omega_\epsilon(p)$. Cover $\text{supp}(\eta)$ with a locally finite collection of such open sets and choose a partition of unity $\{f_i\}$ subordinate to it. It follows from Theorems 4.2.2 and 4.1.6 that $f_J \eta = f_{\sum (m_{f_i} J)} \eta = \sum_i f_{m_{f_i} J} \eta = \sum_i f_J f_i \cdot \eta = 0$. The proof for $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$ and $\eta \in \mathcal{B}_k(\mathbb{R}^n)$ is essentially the same. \square

Definition 5.0.9. *Let $0 \leq k \leq n$ and $r \geq 0$. Let $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ and $X \subseteq \mathbb{R}^n$ a closed set. We say that J is **accessible** in X , if for every open set $W \subseteq \mathbb{R}^n$ containing X , there exists $A_i \in \mathcal{A}_k(W)$ with $A_i \rightarrow J$ in $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$. For $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$ the same definition holds, except that we require $A_i \rightarrow J$ in $\hat{\mathcal{B}}_k(\mathbb{R}^n)$.*

Clearly, every differential chain J is accessible in \mathbb{R}^n .

Theorem 5.0.10. *If J is a differential chain and $X \subset \mathbb{R}^n$ is closed, then J is accessible in X if and only if $\text{supp}(J) \subset X$.*

Proof. Let $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, $r \geq 0$, and suppose J is accessible in X . Suppose $p \in \text{supp}(J)$ and $p \notin X$. Since $p \notin X$, there exists $\epsilon > 0$ with $\overline{B_\epsilon(p)} \cap X = \emptyset$. Since $p \in \text{supp}(J)$, there exists $\eta \in \mathcal{B}_k^r(\mathbb{R}^n)$ with $f_J \eta \neq 0$ and $\text{supp}(\eta) \subset \Omega_\epsilon(p)$. Let U be a neighborhood of X . Then $U - \overline{B_\epsilon(p)}$ is also a neighborhood of X , so there exist $A_i \rightarrow J$ in $U - \overline{B_\epsilon(p)}$. We know $f_{A_i} \eta = 0$ since η is supported in $\Omega_\epsilon(p)$, and thus $f_{A_i} \eta \rightarrow f_J \eta = 0$, a contradiction.

Conversely, suppose $\text{supp}(J) \subset X$. Let W be a neighborhood of X . Then W is a neighborhood of $\text{supp}(J)$. Let $A_i \rightarrow J$ in $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ and $B_i := A_i|_W$. We show $B_i \rightarrow J$ in $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$. Let V be an open set with $\text{supp}(J) \subset V \subset W$. There exists a locally finite cover of V^c by open sets $\{U_i\}_{i=1}^\infty$ satisfying $f_J \omega = 0$ for all ω with $\text{supp}(\omega) \subset U_i$ for all $i \geq 1$. Let $\{f_i, f_0\}_{i=1}^\infty$ be a partition of unity subordinate to $\{U_i, W\}_{i=1}^\infty$, which is a locally finite open cover of \mathbb{R}^n . Let $\omega \in \mathcal{B}_k^r(\mathbb{R}^n)$. Theorems 4.2.2 and 4.1.6 yield $f_J \omega = \sum_{i=0}^\infty f_{m_{f_i} J} \omega = \sum_{i=0}^\infty f_J f_i \cdot \omega = f_J f_0 \omega$. Thus $f_J \omega = f_J f_0 \cdot \omega = \lim_{i \rightarrow \infty} f_{A_i} f_0 \omega = \lim_{i \rightarrow \infty} f_{B_i} \omega$. The result follows since $B_i \in \mathcal{A}_k(W)$.

The proof for $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$ is similar. \square

Proposition 5.0.11. *Let $r \geq 0$ and $0 \leq k \leq n$. The following properties hold:*

- (a) *If a differential chain J is accessible in the closed sets X and Y , then J is accessible in $X \cap Y$.*
- (b) *If the differential chains J and K are accessible in a closed set X , then $J + K$ and cJ are accessible in X for all $c \in \mathbb{R}$.*

Proof. (a): According to Theorem 5.0.10 $\text{supp}(J) \subset X \cap Y$, and thus J is accessible in $X \cap Y$.

(b): Let U be a neighborhood of X . Then there exist $A_i \rightarrow J$ and $B_i \rightarrow K$ with $A_i, B_i \in \mathcal{A}_k^r(U)$. Thus $A_i + B_i \rightarrow J + K$ and $A_i + B_i \in \mathcal{A}_k(U)$. Homogeneity is similar. \square

Corollary 5.0.12. *$\text{supp}(J)$ is the intersection of all closed sets in which J is accessible.*

Proof. Let $E = \bigcap_{\iota \in I} X_\iota$ where the intersection is taken over all closed set X_ι in which J is accessible and I is an indexing set. Let $p \in \text{supp}(J)$. Then $p \in X_\iota$ for each $\iota \in I$ by Theorem 5.0.10. Hence $p \in \bigcap_{\iota \in I} X_\iota$. Thus $\text{supp}(J) \subseteq E$. Let $p \in E$. Then $p \in X_\iota$ for all $\iota \in I$. In particular, $p \in \text{supp}(J)$ since $\text{supp}(J)$ is an accessible set (use Theorem 5.0.10). Hence $E \subseteq \text{supp}(J)$, and we are done. \square

Thus $\text{supp}(J)$ is the smallest closed set in which J is accessible.

Theorem 5.0.13. *Suppose $X \subset \mathbb{R}^n$ is closed and $\text{supp}(J_i) \subset X$ for a sequence of differential chains $J_i \rightarrow J$. Then $\text{supp}(J) \subset X$.*

Proof. If $J = 0$, we are finished since $\text{supp}(J) = \emptyset$ by Definition 5.0.5. Suppose $J \neq 0$. Then $\text{supp}(J) \neq \emptyset$ by Theorem 5.0.7.

Assume $J_i \rightarrow J$ in $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$, $r \geq 0$, or $J_i \rightarrow J$ in $\hat{\mathcal{B}}_k(\mathbb{R}^n)$. Suppose there exists $p \in \text{supp}(J)$ and $p \notin X$. There exists $\epsilon > 0$ such that $\Omega_\epsilon(p) \cap X = \emptyset$, and $\eta \in \mathcal{B}_k^r(\mathbb{R}^n)$ with $\int_J \eta \neq 0$ and $\text{supp}(\eta) \subset \Omega_\epsilon(p)$. On the other hand, $\int_J \eta = \lim_{i \rightarrow \infty} \int_{J_i} \eta = 0$ since $\text{supp}(J_i) \subset X$. The proof for $J_i \rightarrow J$ in $\hat{\mathcal{B}}_k(\mathbb{R}^n)$ is essentially the same. \square

Proposition 5.0.14. *Suppose $J \in \hat{\mathcal{B}}_k^r(\mathbb{R}^n)$ and $f \in \mathcal{B}_0^r(\mathbb{R}^n)$, $r \geq 0$, or $J \in \hat{\mathcal{B}}_k(\mathbb{R}^n)$ and $f \in \mathcal{B}_0(\mathbb{R}^n)$. Then $\text{supp}(m_f J) \subseteq \text{supp}(f) \cap \text{supp}(J)$.*

Proof. Let $p \in \text{supp}(m_f J)$. If $p \notin \text{supp}(f) \cap \text{supp}(J)$, there exists $\epsilon > 0$ with $\Omega_\epsilon(p) \cap \text{supp}(f) \cap \text{supp}(J) = \emptyset$ and $\eta \in \mathcal{B}_k^r(\mathbb{R}^n)$ (respectively, $\eta \in \mathcal{B}_k(\mathbb{R}^n)$) with $\int_J f \cdot \eta = \int_{m_f J} \eta \neq 0$ and $\text{supp}(\eta) \subset \Omega_\epsilon(p)$. Thus $p \in \text{supp}(J)$. If $p \notin \text{supp}(f)$, we can choose $\epsilon > 0$ so that $\Omega_\epsilon(p) \cap \text{supp}(f) = \emptyset$. Then $\int_J f \cdot \eta = 0$, which is a contradiction. Hence $p \in \text{supp}(f) \cap \text{supp}(J)$, as claimed. \square

Proposition 5.0.15. *If T is a continuous operator on $\hat{\mathcal{B}}(\mathbb{R}^n)$ with $\text{supp}(T(A)) \subseteq \text{supp}(A)$ for all $A \in \mathcal{A}(\mathbb{R}^n)$, then $\text{supp}(T(J)) \subseteq \text{supp}(J)$ for all $J \in \hat{\mathcal{B}}(\mathbb{R}^n)$.*

Proof. Let S be the dual operator on $\mathcal{B}(\mathbb{R}^n)$ given by $S\omega := \omega T$. Suppose $p \in \text{supp}(T(J))$ and $p \notin \text{supp}(J)$. There exists $\epsilon > 0$ such that $\Omega_\epsilon(p) \cap \text{supp}(J) = \emptyset$. Then $f_J \omega = 0$ for all $\omega \in \mathcal{B}_k(\mathbb{R}^n)$ supported in $\Omega_\epsilon(p)$. Since $p \in \text{supp}(T(J))$, there exists $\eta \in \mathcal{B}_k(\mathbb{R}^n)$ supported in $\Omega_\epsilon(p)$ so that $f_J S(\eta) = f_{T(J)} \eta \neq 0$. This is a contradiction since $S(\eta) \in \mathcal{B}(\mathbb{R}^n)$. \square

This result also holds for operators $T : \hat{\mathcal{B}}_k^r(\mathbb{R}^n) \rightarrow \hat{\mathcal{B}}_j^s(\mathbb{R}^n)$ with suitable modifications to the subscripts and superscripts.

Examples 5.0.16.

- The unit interval $I = \{(t, 0) \mid 0 \leq t \leq 1\} \subset \mathbb{R}^2$ is the support of uncountably many k -dimensional chains in $\hat{\mathcal{B}}_k^r(\mathbb{R}^2)$ for each $0 \leq k \leq 2$. Examples include $m_f \tilde{I}$, $E_{e_3} \tilde{I}$, $\perp \tilde{I}$, and $P_{e_3} \tilde{I}$ (use Proposition 5.0.15).
- Any closed set S supports a chain. Simply choose a dense countable subset $\{p_i\} \subseteq S$, and note that S supports the chain $J = \sum_{i=1}^\infty (p_i; 1/2^i)$. There are uncountably many distinct chains with support S since $\text{supp}(m_f J) \subset \text{supp}(J)$ by Proposition 5.0.14.

Proposition 5.0.17. *If $W \subset \mathbb{R}^n$ is bounded and open, then $\text{supp}(\widetilde{W}) = \overline{W}$. If σ is an affine k -cell in \mathbb{R}^n , then $\text{supp}(\widetilde{\sigma}) = \overline{\sigma}$.*

Proof. By Theorem 2.9.4 $\widetilde{W} \in \hat{\mathcal{B}}_n^1(\mathbb{R}^n)$. We first show that $\text{supp}(\widetilde{W}) \subseteq \overline{W}$: Suppose $p \in \text{supp}(\widetilde{W})$ and $p \notin \overline{W}$. There exists $\epsilon > 0$ such that $\Omega_\epsilon(p) \cap \overline{W} \neq \emptyset$. Furthermore, there exists $\eta \in \mathcal{B}_k^r(\mathbb{R}^n)$ supported in $\Omega_\epsilon(p)$ with $\int_W \eta = \int_{\widetilde{W}} \eta \neq 0$ by Theorem 2.9.4. But $\int_W \eta = 0$ for all η supported in $\Omega_\epsilon(p)$.

We next show $W \subset \text{supp}(\widetilde{W})$: Suppose there exists $p \in W$ and $p \notin \text{supp}(\widetilde{W})$. Then $f_{\widetilde{W}} \omega = 0$ for all $\omega \in \mathcal{B}_n^1(\mathbb{R}^n)$ supported away from $\text{supp}(\widetilde{W})$. Since $p \in W$, there exists a differential chain $\eta \in \mathcal{B}_n(\mathbb{R}^n)$ supported in $B_\epsilon(p)$, and $\int_W \eta = \int_{\widetilde{W}} \eta \neq 0$ by Theorem 2.9.4.

Now suppose $p \in \overline{W}$ and $p \notin \text{supp}(\widetilde{W})$. There exists $\epsilon > 0$ with $\Omega_\epsilon(p) \cap \text{supp}(\widetilde{W}) = \emptyset$. Therefore, there exists $q \in W \cap \Omega_\epsilon(p)$. It follows that $q \in \text{supp}(\widetilde{W})$, and therefore $\int_{\widetilde{W}} \eta \neq 0$ for some $\eta \in \mathcal{B}_k^r(\mathbb{R}^n)$ supported in $\Omega_\epsilon(p)$. On the other hand, $\int_{\widetilde{W}} \omega = 0$ for all ω supported in $\Omega_\epsilon(p)$.

The result extends readily to affine k -cells by working within the affine subspace containing σ and applying what we just proved. \square

Proposition 5.0.18. *If $W \subset \mathbb{R}^n$ is bounded and open¹⁹, then $\text{supp}(\partial \widetilde{W}) = \text{fr}(W)$.*

Proof. Since W is bounded and open, we know $\text{supp}(\partial \widetilde{W}) \subset \text{supp}(\widetilde{W}) = \overline{W}$ by Propositions 5.0.15 and 5.0.17. Suppose $p \in \text{supp}(\partial \widetilde{W})$ and $p \in W$. Then there exists $\epsilon > 0$ with $\Omega_\epsilon(p) \cap \text{fr}(W) = \emptyset$. There exists $\eta \in \mathcal{B}_{n-1}^r(\mathbb{R}^n)$ with η supported in $\Omega_\epsilon(p)$ and $\int_{\partial \widetilde{W}} \eta \neq 0$. Using Whitney decomposition, say, we can find open sets $W_i \subset W$ with piecewise linear frontiers, $\overline{B_\epsilon(p)} \subset W_{i_0}$, for some i_0 , and $\widetilde{W}_i \rightarrow \widetilde{W}$. By the classical Stokes' Theorem for the Riemann integral of differential forms defined over n -dimensional domains with piecewise linear boundaries, Theorem 2.9.4 and Stokes' Theorem 3.5.4

$$0 = \int_{\partial(W_{i_0})} \eta = \int_{W_{i_0}} d\eta = \int_{\widetilde{W}_{i_0}} d\eta = \int_{\partial \widetilde{W}_{i_0}} \eta \neq 0,$$

contradicting our assumption that $p \in W$. It follows that $p \in \text{fr}(W)$.

¹⁹The author thanks E. Fried and B. Sequin for suggesting this result might be true, as well as a proof for the second half

For the converse, suppose there exists $p \in \text{fr}(W)$ and $p \notin \text{supp}(\partial \widetilde{W})$. There exists $\epsilon > 0$ such that $\Omega_\epsilon(p) \cap \text{supp}(\partial \widetilde{W}) = \emptyset$. Then $\int_{\partial \widetilde{W}} \omega = 0$ for all ω supported in $\Omega_\epsilon(p)$. Let $q \in \Omega_\epsilon(p) \cap W$, which must exist by definition of frontier. Therefore, by Theorems 2.9.4 and 3.5.4, there exists η supported in $\Omega_\epsilon(p)$ with $0 \neq \int_W d\eta = \int_{\widetilde{W}} d\eta = \int_{\partial \widetilde{W}} \eta \neq 0$, contradicting our assumption that $p \notin \text{supp}(\partial \widetilde{W})$. \square

6 The classical integral theorems in open sets

The integral theorems of Stokes, Gauss, and Kelvin-Stokes reach a new level of generality in this section. Here, they hold for matching pairs of differential chains and differential forms defined in open sets $U \subset \mathbb{R}^n$.

We will see that neither differential chains nor differential forms defined in open subsets of \mathbb{R}^n are simply restrictions of differential chains and differential forms defined in \mathbb{R}^n . The new integral theorems presented in this section are therefore not restrictions of our earlier results in §3 to open sets. The differential forms are smooth inside U , but might be discontinuous at points in the frontier of U .

6.1 Differential chains in open sets

Let $U \subseteq \mathbb{R}^n$ be open. We say that a j -difference k -chain $\Delta_{\sigma^j}(p; \alpha)$ is *inside* W if the convex hull of $\text{supp}(\Delta_{\sigma^j}(p; \alpha))$ is contained in W .

Definition 6.1.1.

$$\|A\|_{B^r, U} := \inf \left\{ \sum_{i=1}^m \|\sigma_i^{j(i)}\| \|\alpha_i\| : A = \sum_{i=1}^m \Delta_{\sigma_i^{j(i)}}(p_i; \alpha_i), \Delta_{\sigma_i^{j(i)}}(p_i; \alpha_i) \text{ is inside } U, \text{ and } j(i) \leq r \text{ for all } i \right\}.$$

Theorem 6.1.2. $\|\cdot\|_{B^r, U}$ is a norm on $\mathcal{A}_k(U)$.

The proof is similar to the proof that $\|\cdot\|_{B^r}$ is a norm and we omit it. Using the norm (6.1.1), we complete the free space $\mathcal{A}_k(U)$, and obtain a Banach space $\hat{\mathcal{B}}_k^r(U)$.

Examples 6.1.3.

- (a) If U is a bounded and open subset of \mathbb{R}^n , then $\widetilde{U} \in \hat{\mathcal{B}}_n^1(U)$: This holds for convex bounded open sets Q because $\widetilde{Q} = \lim_{i \rightarrow \infty} A_i$ where $A_i \in \mathcal{A}_n(Q)$ and all difference chains using Dirac chains $\mathcal{A}_k(Q)$ are inside Q . Using Theorem 2.9.4, we know $\widetilde{U} = \sum \widetilde{Q}_i$ in $\hat{\mathcal{B}}_n^1(\mathbb{R}^n)$ where $U = \cup Q_i$ is a Whitney decomposition of U . The same proof, which uses Cauchy sequences, shows that $\sum \widetilde{Q}_i$ also converges in $\hat{\mathcal{B}}_n^1(U)$, since each partial sum is an element of $\hat{\mathcal{B}}_n^1(U)$.
- (b) Let U be the open set \mathbb{R}^2 less the nonnegative x -axis²⁰. The series $\sum_{n=1}^\infty ((n, 1/n^2); 1) + ((n, -1/n^2); -1)$ of Dirac 0-chains converges in $\hat{\mathcal{B}}_0^1(\mathbb{R}^2)$, but not in $\hat{\mathcal{B}}_0^1(U)$. The idea is that the difference chains needed to prove convergence in $\hat{\mathcal{B}}_0^1(\mathbb{R}^2)$ are not inside U .

²⁰There is a helpful discussion of problems encountered when trying to define chains in open sets in [Whi57] Chapter VIII where we found this example. However, we do not follow Whitney's lead who required chains in open sets U to have their supports contained in U .

Definition 6.1.4. Let $\hat{\mathcal{B}}_k(U) := \varinjlim_r \hat{\mathcal{B}}_k^r(U)$, endowed with the inductive limit topology $\tau_k(U)$, and $\hat{\mathcal{B}}(U) := \oplus_{k=0}^n \hat{\mathcal{B}}_k(U)$, endowed with the direct sum topology. Define the **linking maps** $u_k^{r,s} : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k^s(U)$ where $r \leq s$, and the **canonical inclusions** $u_k^r : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k(U)$, as in §2.11. Define the **hull** subspace $H_k(U) \subset \hat{\mathcal{B}}_k(U)$ just as we defined $H_k(\mathbb{R}^n)$ in Definition 2.11.13, except that the domain of definition is $U \subset \mathbb{R}^n$ instead of \mathbb{R}^n .

Theorem 2.11.14 and Corollary 2.11.4 extend to open sets $U \subset U' \subset \mathbb{R}^n$ and have essentially the same proofs, replacing \mathbb{R}^n with U or U' , as appropriate.

Theorem 6.1.5. If $T : \oplus_{k=0}^n \oplus_{r=0}^\infty \hat{\mathcal{B}}_k^r(U) \rightarrow \oplus_{k=0}^n \oplus_{r=0}^\infty \hat{\mathcal{B}}_k^r(U')$ is a continuous bigraded linear map with $T(\hat{\mathcal{B}}_k^r(U)) \subset \hat{\mathcal{B}}_k^s(U')$, and $T(H_k(U)) \subset H_k(U')$, then T factors through a continuous graded linear map $\hat{T} : \hat{\mathcal{B}}(U) \rightarrow \hat{\mathcal{B}}(U')$ with $T = \pi \circ \hat{T}$.

6.2 Dual spaces

Definition 6.2.1. For $\omega \in (\mathcal{A}_k(U))^*$, let $\|\omega\|_{B^r, U} := \sup\{\omega(\Delta_{\sigma^j}(p; \alpha)) / \|\sigma\| \|\alpha\| : \Delta_{\sigma^j}(p; \alpha) \text{ is inside } U\}$. Let $\mathcal{B}_k^r(U) := \{\omega \in (\mathcal{A}_k(U))^* : \|\omega\|_{B^r, U} < \infty\}$.

The Banach space $\mathcal{B}_k^r(U)$ contains all differential forms $\mathcal{B}_k^r(\mathbb{R}^n)$ restricted to U since the definition of the former only considers difference chains inside U . However, not all elements of $\mathcal{B}_k^r(U)$ extend to elements of $\mathcal{B}_k^r(\mathbb{R}^n)$, unless U has a smoothly embedded frontier.

Examples 6.2.2.

- (a) The characteristic function χ_U is an element of $\mathcal{B}_n^r(U)$ for all $r \geq 0$.
- (b) Let $p \in \mathbb{R}^n$ and $\epsilon > 0$. Then $dx|_{B_\epsilon(p)} \in \mathcal{B}_1^r(B_\epsilon(p))$ for all $r \geq 0$.
- (c) Let $W \subset \mathbb{R}^2$ be the open set depicted in Figure 8. It is easy to construct a differential form $\omega \in \mathcal{B}_0(W)$ which is identically one in the shaded region above the point x and identically zero in the shaded region below x . Clearly, ω is not extendable to a neighborhood of W .
- (d) Let Q be the open unit disk in \mathbb{R}^2 . The sequence $((1 - 1/n, 0); e_1 \wedge e_2)$ converges to $((1, 0); e_1 \wedge e_2)$ in both $\hat{\mathcal{B}}_2^1(Q)$ and $\hat{\mathcal{B}}_2^1(\mathbb{R}^2)$. Now $(1, 0) \notin Q$, but $((1, 0); e_1 \wedge e_2)$ is still a chain ready to be acted upon by elements $\omega \in \mathcal{B}_2(Q)$. For example,

$$\chi_Q dx dy((1, 0); e_1 \wedge e_2) = \lim_{n \rightarrow \infty} \chi_Q dx dy((1 - 1/n, 0); e_1 \wedge e_2) = 1.$$

- (e) Let Q' be the unit disk Q in \mathbb{R}^2 minus the closed set $[0, 1] \times \{0\}$, and define

$$\omega_0(x, y) = \begin{cases} \inf\{x, 1\}, & \text{if } 0 < x < 1, 0 < y < 1 \\ 0, & \text{else} \end{cases}.$$

This Lipschitz function $\omega_0 \in \mathcal{B}_0^1(Q')$ is not extendable to a Lipschitz function on \mathbb{R}^2 . (See Figure 9 for a related example.)

Let $\Theta_{k,r} : \mathcal{B}_k^r(U) \rightarrow (\mathcal{A}_k(U))^*$ be the linear map given by $\omega \mapsto \{A \mapsto \omega(A)\}$ for all $A \in \mathcal{A}_k(U)$.

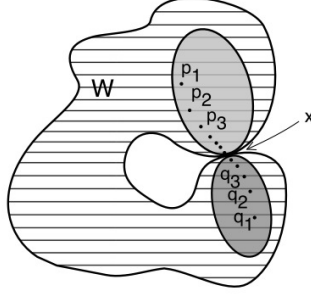


Figure 8: Distinct limit chains supported in x are separated by a form in $\hat{\mathcal{B}}_0(U)$

Theorem 6.2.3. $\Theta_{k,r} : \mathcal{B}_k^r(U) \rightarrow (\hat{\mathcal{B}}_k^r(U))'$ is a topological isomorphism for all $r \geq 1, 0 \leq k \leq n$ and $U \subseteq \mathbb{R}^n$ open and smoothly embedded. Furthermore, $\|\Theta_{k,r}(\omega)\|_{B^r,U} = \|\omega\|_{B^r,U}$ for all $0 \leq r < \infty$.

Proof. Let $\omega \in \mathcal{B}_k^r(U)$. Then ω is extendable to $\tilde{\omega} \in \mathcal{B}_k^r(\mathbb{R}^n)$. The isomorphism $\Theta_r : \mathcal{B}_k^r(\mathbb{R}^n) \rightarrow (\hat{\mathcal{B}}_k^r(\mathbb{R}^n))'$ of Theorem 2.8.2 is continuous. Then $\Theta_r(\tilde{\omega})|_{\hat{\mathcal{B}}_k^r(U)}$ determined by $J \mapsto \tilde{\omega}(J) = \omega(J)$ for $J \in \hat{\mathcal{B}}_k^r(U)$ is an element of $(\hat{\mathcal{B}}_k^r(U))'$, independent of choice of the extension $\tilde{\omega}$. If $\Theta_r(\tilde{\omega})|_{\hat{\mathcal{B}}_k^r(U)} = 0$, it follows that $\omega(J) = 0$ for all $J \in \hat{\mathcal{B}}_k^r(U)$. Thus $\omega = 0$. Since $\Theta_{k,r} = \Theta_r(\tilde{\omega})|_{\hat{\mathcal{B}}_k^r(U)}$, this proves that $\Theta_{k,r}$ is injective. Surjectivity is straightforward: Given $X \in (\hat{\mathcal{B}}_k^r(U))'$, define $\omega(A) = X(A)$ for all $A \in \mathcal{A}_k(U)$. It is not hard to show that $\|\omega\|_{B^r,U} = \|X\|_{B^r,U}$ by only considering difference chains inside A . It follows that $\Theta_{k,r}$ is continuous. \square

Definition 6.2.4. Let $\mathcal{B}_k(U) := \varprojlim \mathcal{B}_k^r(U)$ endowed with the projective limit topology, making it into a Fréchet space.

Theorem 6.2.5. The topological vector space of differential k -cochains $(\hat{\mathcal{B}}_k(U))'$ is isomorphic to the Fréchet space of differential k -forms $\mathcal{B}_k(U)$.

The proof is the same as that for Theorem 2.11.12.

Definition 6.2.6. Suppose $J \in \hat{\mathcal{B}}_k^r(U)$, $\omega \in \mathcal{B}_k^r(U)$, and $r \geq 0$, or $J \in \hat{\mathcal{B}}_k(U)$ and $\omega \in \mathcal{B}_k(U)$. Define $\int_J \omega := \lim_{i \rightarrow \infty} \omega(A_i)$ where $A_i \rightarrow J$ in the appropriate topology (see 6.2.5).

Theorem 6.2.7. The integral pairing is separately continuous.

6.3 Inclusions $\mathcal{A}_k(U) \hookrightarrow \hat{\mathcal{B}}_k^r(U)$

Definition 6.3.1. Suppose $U \subseteq U' \subseteq \mathbb{R}^n$, $0 \leq k \leq n$, and $r \geq 0$. Define $\psi_k^r : \mathcal{A}_k(U) \hookrightarrow \mathcal{A}_k(U')$ by $\psi_k^r := \text{Id}$. Thus $\psi_k^r(p; \alpha) = (p; \alpha)$ for all k -elements $(p; \alpha) \in \mathcal{A}_k(U)$.

Lemma 6.3.2. Suppose $U \subseteq U' \subseteq \mathbb{R}^n$. If $A \in \mathcal{A}_k(U)$, then $\|A\|_{B^r,U'} \leq \|A\|_{B^r,U}$ for all $r \geq 0$.

Proof. This follows since the definition of $\|A\|_{B^r,U'}$ considers more difference chains than does that of $\|A\|_{B^r,U}$. \square

Theorem 6.3.3. *The linear map ψ_k^r extends to a continuous linear map $\psi_k^r : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k^r(U')$ for each $r \geq 0$ and $0 \leq k \leq n$ satisfying $\|J\|_{B^{r,U'}} \leq \|J\|_{B^{r,U}}$ for all $J \in \hat{\mathcal{B}}_k^r(U)$. For each $k \geq 0$, there exists a continuous linear map $\psi_k : \hat{\mathcal{B}}_k(U) \rightarrow \hat{\mathcal{B}}_k(U')$ satisfying $\psi_k \circ u_k^r = \psi_k^r$ for all $r \geq 0$. However, ψ_k^r and ψ_k are not injective unless U is smoothly embedded in U' .*

Proof. The inequality follows from Lemma 6.3.2 and density of $\mathcal{A}_k(U)$ in $\hat{\mathcal{B}}_k^r(U)$. The extension to the inductive limit follows since the hull condition holds: $\psi_k^r(H_k(U)) \subset H_k(U')$. We can therefore apply Theorem 2.11.14. Example 1 shows that these maps are not generally injective. \square

Definition 6.3.4. *Define the linear map $\psi_{k,r} : \hat{\mathcal{B}}_k^r(U_1) \rightarrow \hat{\mathcal{B}}_k^r(U_2)$ of Banach spaces as follows: Suppose $J \in \hat{\mathcal{B}}_k^r(U_1)$. Then there exist Dirac chains $A_i \rightarrow J$ in the B^{r,U_1} norm. By Lemma 6.3.2 $\{A_i\}$ forms a Cauchy sequence in the B^{r,U_2} norm. Let $\psi_{k,r}(J) = \lim_{i \rightarrow \infty} A_i$ in the B^{r,U_2} norm. The continuous linear map $\psi_{k,r}$ extends to continuous linear maps $\psi_k : \hat{\mathcal{B}}_k^\infty(U_1) \rightarrow \hat{\mathcal{B}}_k^\infty(U_2)$, and $\psi_k : \hat{\mathcal{B}}_k(U_1) \rightarrow \hat{\mathcal{B}}_k(U_2)$.*

6.4 Support of a chain in an open set

Definition 6.4.1. *Suppose $U \subseteq \mathbb{R}^n$ is open. If $J \in \hat{\mathcal{B}}_k^r(U)$, let*

$$\text{supp}(J) := \left\{ p \in U : \text{for all } \epsilon > 0, \text{ there exists } \eta \in \mathcal{B}_k^r(U) \text{ s.t. } \int_J \eta \neq 0 \text{ where } \text{supp}(\eta) \subset \Omega_\epsilon(p) \right\}.$$

If $J \in \hat{\mathcal{B}}_k(U)$, let

$$\text{supp}(J) := \left\{ p \in U : \text{for all } \epsilon > 0, \text{ there exists } \eta \in \mathcal{B}_k(U) \text{ s.t. } \int_J \eta \neq 0 \text{ where } \text{supp}(\eta) \subset \Omega_\epsilon(p) \right\}.$$

If U is smoothly embedded in \mathbb{R}^n , then the support of $J \in \hat{\mathcal{B}}_k^r(U)$ coincides with its support when included in $\hat{\mathcal{B}}_k^r(\mathbb{R}^n)$.

Proposition 6.4.2. *The subset of all chains in $\hat{\mathcal{B}}_k^r(U)$ that are supported in U is a proper subspace of $\hat{\mathcal{B}}_k^r(U)$.*

Proof. Observe $\text{supp}(0) = \emptyset \subset U$. Let $J, K \in \hat{\mathcal{B}}_k^r(U)$ with $\text{supp}(J) \cup \text{supp}(K) \subset U$. Since $J + K \in \hat{\mathcal{B}}_k^r(U)$, then $\text{supp}(J + K) \subseteq \overline{U}$ by Definition 6.4.1. Let $p \in \text{supp}(J + K)$. Suppose $p \in \text{fr}(U)$. Choose $\epsilon > 0$ so small that $\Omega_\epsilon(p) \cap (\text{supp}(J) \cup \text{supp}(K)) = \emptyset$. Then there exists $\eta \in \mathcal{B}_k^r(U)$ with $\text{supp}(\eta) \subset \Omega_\epsilon(p)$ such that $0 = \int_J \eta + \int_K \eta = \int_{J+K} \eta \neq 0$, contradicting our assumption that $p \in \text{fr}(U)$. Thus $p \in U$. The subspace is proper since $\text{supp}(\tilde{U}) = \overline{U}$ by Proposition 5.0.17. Homogeneity is easier and we omit it. \square

Theorem 6.4.3. *Suppose $U \subset U' \subset \mathbb{R}^n$. The linear maps $\psi_{k,r} : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k^r(U')$ and $\psi_k : \hat{\mathcal{B}}_k(U) \rightarrow \hat{\mathcal{B}}_k(U')$ are injections if U is smoothly embedded in U' , or on restriction to the subspace of chains supported in U .*

Proof. Suppose $J \in \hat{\mathcal{B}}_k^r(U)$ satisfies $\psi_{k,r}(J) = 0$. It suffices to show that $\int_J \omega = 0$ for all $\omega \in \mathcal{B}_k^r(U)$, for this will show $J = 0$. Since U is smoothly embedded, each ω extends to a differential form $\eta \in \mathcal{B}_k^r(U')$, and hence $\int_{\psi_{k,r}(J)} \eta = 0$. Let $A_i \rightarrow J$ in $\hat{\mathcal{B}}_k^r(U)$. Then $\psi_{k,r}(A_i) \rightarrow \psi_{k,r}(J)$ by Lemma 6.3.2. Since $\psi_{k,r}$ is the identity on Dirac chains, $\int_J \omega = \lim_{i \rightarrow \infty} \int_{A_i} \omega = \lim_{i \rightarrow \infty} \int_{\psi_{k,r}(A_i)} \eta = \int_{\psi_{k,r}(J)} \eta = 0$. \square

The maps ψ are not generally injective unless U is embedded.

The results of §5 carry over to $\hat{\mathcal{B}}_k^r(U)$ and $\hat{\mathcal{B}}_k(U)$. We state a few of the general versions here for reference, but without proof.

Proposition 6.4.4. *Let $J \in \hat{\mathcal{B}}_k^r(U)$. Then $\text{supp}(J) = \text{supp}(u_k^{r,s}(J)) = \text{supp}(u_k^r(J))$ for all $0 \leq r \leq s$.*

Theorem 6.4.5. *If J is a nonzero differential chain in U , then $\text{supp}(J)$ is a closed, nonempty set.*

Definition 6.4.6. *Let $0 \leq k \leq n$ and $r \geq 0$. Let $J \in \hat{\mathcal{B}}_k^r(U)$ and $X \subseteq U$ a closed set. We say that J is **accessible** in U , if for every open set $W \subseteq U$ containing X , there exists $A_i \in \mathcal{A}_k(W)$ with $A_i \rightarrow J$ in $\hat{\mathcal{B}}_k^r(U)$. For $J \in \hat{\mathcal{B}}_k(U)$ the same definition holds, except that we require $A_i \rightarrow J$ in $\hat{\mathcal{B}}_k(U)$.*

Theorem 6.4.7. *If J is a differential chain in U and $X \subset U$ is closed, then J is accessible in X if and only if $\text{supp}(J) \subset X$.*

Theorem 6.4.8. *If J is a differential chain in an open set U , then $\text{supp}(J)$ can be written as the intersection of all closed sets in which J is accessible.*

Theorem 6.4.9. *Suppose $X \subset U$ is closed and $\text{supp}(J_i) \subset X_0$ for a sequence of chains $J_i \rightarrow J$ in $\hat{\mathcal{B}}_k^r(U)$, $r \geq 0$, or $J_i \rightarrow J$ in $\hat{\mathcal{B}}_k(U)$. Then $\text{supp}(J) \subset X_0$.*

Proposition 6.4.10. *Suppose $J \in \hat{\mathcal{B}}_k^r(U)$, $f \in \mathcal{B}_0^r(U)$ and $r \geq 0$, or $J \in \hat{\mathcal{B}}_k(U)$ and $f \in \mathcal{B}_0(U)$. Then $\text{supp}(m_f J) \subseteq \text{supp}(f) \cap \text{supp}(J)$.*

Proposition 6.4.11. *If T is a continuous operator on $\hat{\mathcal{B}}(U)$ with $\text{supp}(T(A)) \subseteq \text{supp}(A)$ for all $A \in \mathcal{A}(U)$, then $\text{supp}(T(J)) \subseteq \text{supp}(J)$ for all $J \in \hat{\mathcal{B}}(U)$.*

Examples 6.4.12.

(a) In Figure 8 there are two sequences of points p_i, q_i converging to x in \mathbb{R}^2 . We know both $(p_i; 1) \rightarrow (x; 1)$ and $(q_i; 1) \rightarrow (x; 1)$ in $\hat{\mathcal{B}}_0^1(\mathbb{R}^2)$. Each of the sequences $\{(p_i; 1)\}$ and $\{(q_i; 1)\}$ is Cauchy in $\hat{\mathcal{B}}_0^1(U)$ since the intervals connecting p_i and p_j , and those connecting q_i and q_j are subsets of U . Therefore $(p_i; 1) \rightarrow A_p$ and $(q_i; 1) \rightarrow A_q$ in $\hat{\mathcal{B}}_0^1(U)$. However, the interval connecting p_i and q_i is not a subset of U . The chains $A_p \neq A_q$ in $\hat{\mathcal{B}}_0^1(U)$ since they are separated by elements of the dual Banach space $\mathcal{B}_0^1(U)$. For example, let ω_1 be a smooth form defined on U which is identically one in the lighter shaded elliptical region, and identically zero in the darker shaded region. Then $\int_{A_p} \omega_1 = \lim_{i \rightarrow \infty} \omega_1(p_i; 1) = 1$ and $\int_{A_q} \omega_1 = \lim_{i \rightarrow \infty} \omega_1(q_i; 1) = 0$. Since $\psi_{0,1}(A_p) = \psi_{0,1}(A_q) = (x; 1)$, then the linear map $\psi_{0,1}$ is not injective.

(b) If U is bounded and open, then $\partial(\bar{U})$ is an element of $\hat{\mathcal{B}}_{n-1}(U)$ by Proposition 5.0.15. Its support is $\text{fr}(U)$. (see Proposition 5.0.18).

6.5 Pushforward and change of variables

Definition 6.5.1. *Suppose $U \subseteq \mathbb{R}^n$ and $U' \subseteq \mathbb{R}^m$ are open, and $F : U \rightarrow U'$ is a differentiable map. For $p \in U$, and $1 \leq k \leq n$, define **linear pushforward** $F_{p*}(v_1 \wedge \cdots \wedge v_k) := DF_p(v_1) \wedge \cdots \wedge DF_p(v_k)$ where DF_p is the total derivative of F at p . For $k = 0$, set $F_{p*}(m) := m$. Define pushforward $F_*(p; \alpha) := (F(p), F_{p*}\alpha)$ for all simple k -elements $(p; \alpha)$ and extend to a linear map $F_* : \mathcal{A}_k(U) \rightarrow \mathcal{A}_k(U')$.*

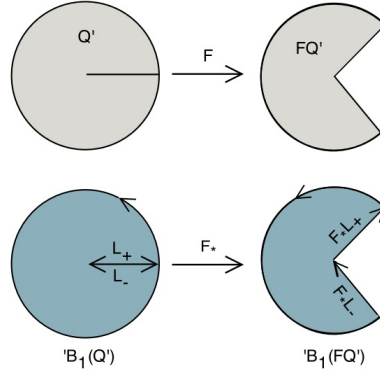


Figure 9: A smooth map F that is not extendable to a neighborhood of $\overline{Q'}$, and nevertheless determines a well-defined pushforward map $F_* : \hat{\mathcal{B}}_1(Q') \rightarrow \hat{\mathcal{B}}_1(F(Q'))$

The classical definition of *pullback* F^* satisfies the relation $F^*\omega(p; \alpha) = \omega(F(p); F_*\alpha) = \omega F_*(p; \alpha)$ for all differentiable maps $F : U \rightarrow U'$, exterior forms $\omega \in (\mathcal{A}_k(U'))^*$, and simple k -elements $(p; \alpha)$ with $p \in U$.

Definition 6.5.2. For $r \geq 1$, let $\mathcal{M}^r(U, \mathbb{R}^m)$ be the vector space of differentiable maps $F : U \rightarrow \mathbb{R}^m$ whose coordinate functions $F_{e_i}(p) := \langle F(p), e_i \rangle$ have directional derivatives in $\mathcal{B}_0^{r-1}(U)$. Let $\mathcal{M}^\infty(U, \mathbb{R}^m)$ be the vector space of differentiable maps $F : U \rightarrow \mathbb{R}^m$ whose coordinate functions $F_{e_i}(p) := \langle F(p), e_i \rangle$ have directional derivatives in $\mathcal{B}_0^{r-1}(U)$ for all $r \geq 1$.

For example, the identity map $x \mapsto x$ is an element of $\mathcal{M}^r(U, \mathbb{R})$. Similarly, $\mathcal{M}^r(U, \mathbb{R})$ includes the coordinate maps $x \mapsto x_i$ and $\mathcal{B}_0^r(U)$ does not, since coordinate maps are not bounded. A 0-form $f_0 \in \mathcal{B}_0^r(U)$ trivially determines a map $f(x) := f_0(x; 1)$ in $\mathcal{M}^r(U, \mathbb{R})$. Thus the space $\mathcal{B}_0^r(U)$ is a proper subset of $\mathcal{M}^r(U, \mathbb{R})$, and the norm on $\mathcal{B}_0^r(U)$ does not extend to $\mathcal{M}^r(U, \mathbb{R})$.

Definition 6.5.3. For $1 \leq r < \infty$, define a seminorm on $\mathcal{M}^r(U, \mathbb{R}^m)$ by

$$\rho_r(F) := \max_{i,j} \{ \|L_{e_j} F_{e_i}\|_{B^{r-1}, U} \}.$$

Endow $\mathcal{M}^r(U, \mathbb{R}^m)$ with the topology μ_U^r induced by this seminorm. That is, μ_U^r is the coarsest topology on $\mathcal{M}^r(U, \mathbb{R}^m)$ such that each map $F \mapsto \rho_r(F - F_0)$ where $F_0 \in \mathcal{M}^r(U, \mathbb{R}^m)$ is continuous. (This is similar to the compact-open topology, but this way we can avoid polynomial mappings beyond the identity.) A base of neighborhoods of $F_0 \in \mathcal{M}^r(U, \mathbb{R}^m)$ for this topology is obtained in the following way: for every $\epsilon > 0$, let

$$U_{r,\epsilon}(F_0) = \{F \in \mathcal{M}^r(U, \mathbb{R}^m) : \rho_r(F - F_0) < \epsilon\}.$$

That the vector space operations are continuous in this topology follows from the definition of a seminorm. The resulting topological vector space is locally convex because each $U_{r,\epsilon}(0)$ is absolutely convex and absorbent. Similarly, create a base of neighborhoods of $F_0 \in \mathcal{M}^\infty(U, \mathbb{R}^m)$ by using $U_{r,\epsilon}(F_0)$ for every $r \geq 1$ and every $\epsilon > 0$.

Lemma 6.5.4. If $F \in \mathcal{M}^r(U, U')$, $0 \leq k \leq n$, and $1 \leq r < \infty$, then

$$\frac{\|F^*\omega\|_{B^{r,U}}}{\|\omega\|_{B^{r,U'}}} \leq n2^r \max\{1, \rho_r(F)\}$$

for all $\omega \in \mathcal{B}_k^r(U')$.

Proof. For $k = 0$, the proof follows using the product and chain rules. The general result follows by using coordinates. (The actual constant $n2^r$ which we obtained is not important, only that it is finite.) \square

Let $\mathcal{M}^r(U, U') \subset \mathcal{M}^r(U, \mathbb{R}^m)$ be the subset of maps whose codomains are open sets $U' \subset \mathbb{R}^m$.

Theorem 6.5.5. *If $F \in \mathcal{M}^r(U, U')$, $0 \leq k \leq n$, and $1 \leq r < \infty$, then*

$$\|F_*A\|_{B^r, U} \leq n2^r \max\{1, \rho_r(F)\} \|A\|_{B^r, U}. \quad (6.1)$$

for all $A \in \mathcal{A}_k(U)$.

Suppose $J \in \hat{\mathcal{B}}_k^r(U)$. Then $J = \lim_{i \rightarrow \infty} A_i$ for some $A_i \in \mathcal{A}_k(U)$. Since $\{A_i\}_i$ is a Cauchy sequence, so is $\{F_*A_i\}_i$ by Theorem 6.5.5. Define $F_*J := \lim_{i \rightarrow \infty} F_*A_i$.

Theorem 6.5.6. *Let $F \in \mathcal{M}^r(U, U')$, $0 \leq k \leq n$, and $r \geq 1$. The linear maps $F_* : \mathcal{A}_k(U) \rightarrow \mathcal{A}_k(U')$ extend to continuous linear maps $F_* : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k^r(U')$ with*

$$\|F_*J\|_{B^r, U} \leq n2^r \max\{1, \rho_r(F)\} \|J\|_{B^r, U}.$$

Furthermore, for each $0 \leq k \leq n$, there exists a unique continuous operator $F_* : \hat{\mathcal{B}}_k(U) \rightarrow \hat{\mathcal{B}}_k(U')$ which restricts to F_{k*} on each $u_k^r(\hat{\mathcal{B}}_k^r(U))$.

Proof. Let $J \in \hat{\mathcal{B}}_k^r(U)$. By Theorem (2.1) and Lemma 6.5.4,

$$\begin{aligned} \|F_*J\|_{B^r} &= \sup_{0 \neq \omega \in \mathcal{B}_k^r} \frac{|\int_J F^* \omega|}{\|\omega\|_{B^r}} \\ &\leq \sup_{0 \neq \omega \in \mathcal{B}_k^r} \frac{\|F^* \omega\|_{B^r, U}}{\|\omega\|_{B^r}} \|J\|_{B^r, U} \leq C \max\{1, \rho_r(F)\} \|J\|_{B^r, U}. \end{aligned}$$

If $F(U) \subset U' \subset \mathbb{R}^m$, then $F_*(H_k(U)) \subset H_k(U')$, so we may apply Theorem 6.1.5 to uniquely extend the F_{k*} to $F_* : \hat{\mathcal{B}}_k(U) \rightarrow \hat{\mathcal{B}}_k(U')$. \square

Example 6.5.7. *Suppose M is a smoothly embedded surface in \mathbb{R}^3 . Let $F : M \rightarrow S^2$ be the Gauss map. Then F is as smooth as M and extends to a smooth map in a neighborhood U of M . Then $F : U \rightarrow U'$ where U' is an open neighborhood of S^2 in \mathbb{R}^3 . The shape operator is given by $u \mapsto F_*u$ where $u \in TM$. Thus the shape operator is the pushforward operator F_* given by the Gauss map F .*

It is straightforward to see that if $F \in \mathcal{M}^r(U_1, U_2)$, $G \in \mathcal{M}^r(U_2, U_3)$, then $G_* \circ F_* = (G \circ F)_*$.

Corollary 6.5.8 (Change of variables). *Suppose $F \in \mathcal{M}^r(U, U')$ and $0 \leq k \leq n$. Then*

$$\int_{F_*J} \omega = \int_J F^* \omega \quad (6.2)$$

for all matching pairs $J \in \hat{\mathcal{B}}_k^r(U)$, $\omega \in \mathcal{B}_k^r(U')$, and $1 \leq r \leq \infty$.

Proof. This follows from Theorems 6.5.6 and 2.8.1. \square

Compare [Lax99] [Lax01] which is widely considered to be the natural change of variables for multivariables. The differential forms version of change of variables holds for C^1 diffeomorphisms F and domains of bounded open sets. Our result presents a coordinate free version in arbitrary dimension and codimension ²¹.

Proposition 6.5.9. *If $F \in \mathcal{M}^r(U, U')$, where F is a closed map, then $\text{supp}(F_*J) \subseteq F(\text{supp}(J)) \subseteq U'$ for all $J \in \hat{\mathcal{B}}_k(U)$ supported in U , and $\text{supp}(F_*J) \subseteq F(\text{supp}(J)) \subseteq \overline{U'}$ for all $J \in \hat{\mathcal{B}}_k(U)$.*

Proof. Suppose $\text{supp}(J) \subset U$. We show that F_*J is accessible in the closed set $F(\text{supp}(J))$. Let N be a neighborhood of $F(\text{supp}(J))$ in U' . Then $F^{-1}N \subseteq U$ is a neighborhood of $\text{supp}(J)$. Since J is accessible in $\text{supp}(J)$, by definition of support, there exists $A_i \rightarrow J$ where $A_i \in \mathcal{A}_k(U)$ is supported in $F^{-1}N$. Therefore $F_*(A_i) \rightarrow F_*J$ and $F_*(A_i) \in \mathcal{A}_k(U')$ is supported in N , showing that F_*J is accessible in $F(\text{supp}(J))$. Hence $\text{supp}(F_*J) \subseteq F(\text{supp}(J))$. \square

6.6 Multiplication by a function in an open set

The next results are straightforward extensions of Theorem 4.1.4 and Theorem 4.1.6.

Given $f : U \rightarrow \mathbb{R}$ where $U \subseteq \mathbb{R}^n$ is open, define the linear map $m_f : \mathcal{A}_k(U) \rightarrow \mathcal{A}_k(U)$ by $m_f(p; \alpha) := f(p)(p; \alpha)$ where $(p; \alpha)$ is an arbitrary k -element with $p \in U$.

Theorem 6.6.1. *Let U be open in \mathbb{R}^n , $0 \leq k \leq n$, and $r \geq 0$. The bilinear map $m = m_k^r : \mathcal{B}_0^r(U) \times \mathcal{A}_k(U) \rightarrow \mathcal{A}_k(U)$ extends to a separately continuous bilinear map $m = m_k^r : \mathcal{B}_0^r(U) \times \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k^r(U)$ with*

$$\|m(f, J)\|_{B^r, U} \leq n2^r \|f\|_{B^r, U} \|J\|_{B^r, U} \text{ for all } J \in \hat{\mathcal{B}}_k^r(U).$$

Furthermore, for each $0 \leq k \leq n$, there exists a separately continuous bilinear map $m = m_k : \mathcal{B}_0(U) \times \hat{\mathcal{B}}_k(U) \rightarrow \hat{\mathcal{B}}_k(U)$ which restricts to m_k^r on each $\mathcal{B}_0^r(U) \times \mathcal{A}_k^r(\hat{\mathcal{B}}_k^r(U))$.

Theorem 6.6.2 (Change of density in open sets). *Let U be open in \mathbb{R}^n and $0 \leq k \leq n$. The space of differential chains $\hat{\mathcal{B}}(U)$ is a graded module over the ring $\mathcal{B}_0(U)$ satisfying*

$$\int_{m_f J} \omega = \int_J f \cdot \omega \tag{6.3}$$

for all matching pairs $J \in \hat{\mathcal{B}}_k^r(U)$, $\omega \in \mathcal{B}_k^r(U)$, and $0 \leq r \leq \infty$.

We remark that a 0-form $f_0 \in \mathcal{B}_0^r(U)$ trivially determines a map $f(x) := f_0(x; 1)$ in $\mathcal{M}^r(U, \mathbb{R})$, a multiplication operator $m_f(p; \alpha) := (p; f(p)\alpha)$ in the ring $\mathcal{B}_0^r(U)$, and a linear map (pushforward) $f_* : \hat{\mathcal{B}}_0^r(U) \rightarrow \hat{\mathcal{B}}_0^r(\mathbb{R})$ with $f_*(p; \alpha) = (f(p); f_*\alpha)$.

²¹Whitney defined the pushforward operator F_* on polyhedral chains and extended it to sharp chains in [Whi57]. He proved a change of variables formula (6.2) for Lipschitz forms. However, the important relation $F_*\partial = \partial F_*$ does not hold for sharp chains since ∂ is not defined for the sharp norm. (See Corollary 8.5.2 below.) The flat norm of Whitney does have a continuous boundary operator, but flat forms are highly unstable. The following example modifies an example of Whitney found on p. 270 of [Whi57] which he used to show that components of flat forms may not be flat. But the same example shows that the flat norm has other problems. The author includes mention of these problems of the flat norm since they are not widely known, and she has seen more than one person devote years trying to develop calculus on fractals using the flat norm. Whitney's great contributions to analysis and topology are not in question, and his promotion of Mackey's idea that "chains come first" was right. **Example:** In \mathbb{R}^2 , let $\omega_t(x, y) = \begin{cases} e_1 + e_2 + tu, & x + y < 0 \\ 0, & x + y > 0 \end{cases}$ where $t \geq 0$ and $u \in \mathbb{R}^2$ is nonzero. Then ω_0 is flat, but ω_t is not flat for any $t > 0$. In particular, setting $t = 2, u = -e_2$, we see that $\star\omega_0$ is not flat.

6.7 The boundary operator and Stokes' Theorem in open sets

Theorem 6.7.1. *The linear map $\partial : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k-1}^{r+1}(U)$ is continuous with $\|\partial J\|_{B^{r+1}} \leq kn\|J\|_{B^r}$ for all $J \in \hat{\mathcal{B}}_k^r(U)$. It therefore extends to a continuous operator $\partial : \hat{\mathcal{B}}(U) \rightarrow \hat{\mathcal{B}}(U)$ and restricts to the chainlet complex $Ch_k^s(U)$.*

Proof. It suffices to prove this for $A \in \mathcal{A}_k(U)$. But $A \in \mathcal{A}_k(\mathbb{R}^n)$, and thus $\|\partial A\|_{B^{r+1}} \leq kn\|A\|_{B^r}$ by Theorem 3.5.2. \square

Theorem 6.7.2 (Stokes' theorem in open sets). *Let $0 \leq k \leq n-1$. Then*

$$\int_{\partial J} \omega = \int_J d\omega$$

for all matching pairs $J \in \hat{\mathcal{B}}_{k+1}^{r-1}(U)$, $\omega \in \mathcal{B}_k^r(U)$, and $1 \leq r \leq \infty$.

Proof. Let $J \in \hat{\mathcal{B}}_{k+1}^{r-1}(U)$ and $\omega \in \mathcal{B}_k^r(U)$. Then $J = \lim_{i \rightarrow \infty} A_i$ where $A_i \in \mathcal{A}_k(U)$. Since $A_i \in \mathcal{A}_k(U)$, we may apply Theorem 3.5.4 to conclude that $\int_{\partial A_i} \omega = \int_{A_i} d\omega$. Now $\omega \in \mathcal{B}_k^r(U)$ implies $d\omega \in \mathcal{B}_{k+1}^{r-1}(U)$. By Theorem 6.7.1

$$\int_{\partial J} \omega = \lim_{i \rightarrow \infty} \int_{\partial A_i} \omega = \lim_{i \rightarrow \infty} \int_{A_i} d\omega = \int_J d\omega.$$

\square

Theorem 6.7.3. *If U is a bounded and open subset of \mathbb{R}^n , then*

- (a) $\tilde{U} \in \hat{\mathcal{B}}_n^1(U)$;
- (b) $\partial \tilde{U} \in \hat{\mathcal{B}}_{n-1}^2(U)$;
- (c) $\int_{\tilde{U}} \omega = \int_U \omega$ for all $\omega \in \mathcal{B}_n^1(U)$.

Proof. Part (a) was established in Example 1.

(b): To show $\partial \tilde{Q} \in \hat{\mathcal{B}}_{n-1}^2(Q)$, it suffices to show that $\partial(p; \alpha) \in \hat{\mathcal{B}}_{n-2}^2(Q)$ for all $p \in Q$ since $\partial \tilde{Q} = \lim_{i \rightarrow \infty} \partial A_i$ where $A_i \in \mathcal{A}_n(Q)$. But this holds since difference chains inside Q can be used to approximate $\partial(p; \alpha)$.

(c): Using Whitney decomposition $U = \cup_{i=1}^{\infty} Q_i$ of Theorem 2.9.4, we have $\int_{\tilde{U}} \omega = \lim_{N \rightarrow \infty} \sum_{i=1}^N \int_{\tilde{Q}_i} \omega = \lim_{N \rightarrow \infty} \sum_{i=1}^N \int_{Q_i} \omega = \int_U \omega$ for all $\omega \in \mathcal{B}_n^1(U)$. The last integral is the Riemann integral. \square

Proposition 6.7.4. *Suppose $F \in \mathcal{M}^r(U, U')$. Then $F_* \partial = \partial F_* : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k-1}^{r+1}(U')$.*

Proof. This follows directly from the dual result on differential forms $F^*d = F^*d$. A direct calculation is possible, but takes longer. \square

Example 6.7.5. Let Q' be the open unit disk Q in \mathbb{R}^2 , less the interval $[0, 1] \times \{0\}$. It follows that $\widetilde{Q}' = \widetilde{Q} \in \hat{\mathcal{B}}_2^1(\mathbb{R}^2)$ since evaluations by n -forms in $\mathcal{B}_n(\mathbb{R}^2)$ are the same over Q' and Q . But $\widetilde{Q}'_{Q'} \in \hat{\mathcal{B}}_2^1(Q')$ and is simply not the same chain as \widetilde{Q}' . Not only are these chains in different topological vector spaces, but their boundaries are qualitatively different. Let $L_+ \in \hat{\mathcal{B}}_1^2(Q')$ be the 1-chain representing the interval $[0, 1] \times \{0\}$ found by approximating $[0, 1] \times \{0\}$ with intervals $[0, 1] \times \{1/n\} \cap Q$. Similarly, let $L_- \in \hat{\mathcal{B}}_1^2(Q')$ be the 1-chain representing the interval $[0, 1] \times \{0\}$ found by approximating $[0, 1] \times \{0\}$ with intervals $[0, 1] \times \{-1/n\}$. Recall ω_0 as defined in Example 5. Then $\omega_0^2 dx \in \mathcal{B}_1^2(Q')$, $\int_{L_+} \omega_0^2 dx = \lim_{n \rightarrow \infty} \int_{[0, 1] \times \{1/n\}} \omega_0^2 dx = 1$ and $\int_{L_-} \omega_0^2 dx = \lim_{n \rightarrow \infty} \int_{[0, 1] \times \{-1/n\}} \omega_0^2 dx = 0$. It follows that $L_+ - L_- \neq 0$ and thus $\partial(\widetilde{Q}'_{Q'}) = L_+ - L_- + \widetilde{S}^1_{Q'} \neq \widetilde{S}^1_{Q'}$ while $\partial\widetilde{Q}' = \widetilde{S}^1$ (see Figure 9).

Proposition 6.7.6. If $F \in \mathcal{M}^r(U, U')$ is a diffeomorphism onto its image, and σ is an affine k -cell in U , then $F_*\widetilde{\sigma} = \widetilde{F\sigma}$, $\text{supp}(F_*\widetilde{\sigma}) = F\sigma$, and $\text{supp}(\partial F_*\widetilde{\sigma}) = F\text{fr}(\sigma)$.

Proof. Since F is a diffeomorphism, $\int_{\widetilde{\sigma}} F^*\omega = \int_{F\sigma} \omega$ from the Cartan theory. Therefore, by Theorem 2.9.4 and $\int_{F_*\widetilde{\sigma}} \omega = \int_{\widetilde{\sigma}} F^*\omega = \int_{\sigma} F^*\omega = \int_{F\sigma} \omega = \int_{\widetilde{F\sigma}} \omega$. \square

6.8 The perpendicular complement operator in open sets

Theorem 6.8.1 (Star theorem in open sets). Let $0 \leq k \leq n$. The linear map $\perp: \mathcal{A}_k(U) \rightarrow \mathcal{A}_{n-k}(U)$ determined by $(p; \alpha) \mapsto (p; \perp \alpha)$, for simple k -elements $(p; \alpha)$, satisfies

$$\|\perp A\|_{B^r} = \|A\|_{B^r}$$

for all $A \in \mathcal{A}_k(U)$ and $r \geq 0$. It therefore extends to a continuous linear map $\perp: \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{n-k}^r(U)$ for each $0 \leq k \leq n, r \geq 0$, and to a continuous graded operator $\perp \in \mathcal{L}(\hat{\mathcal{B}})$ satisfying

$$\int_{\perp J} \omega = \int_J \star \omega$$

for all matching pairs $J \in \hat{\mathcal{B}}_k^r(U)$, $\omega \in \mathcal{B}_{n-k}^r(U)$, and $0 \leq r \leq \infty$.

Proof. This follows from Theorem 3.6.3 since $\mathcal{A}_k(U) \subset \mathcal{A}_k(\mathbb{R}^n)$. \square

6.9 Gauss-Green, and Kelvin-Stokes' Theorems in open sets

Corollary 6.9.1 (Gauss-Green theorem in open sets). Let $1 \leq k \leq n$. Then

$$\int_{\perp \partial J} \omega = \int_J d \star \omega$$

for all matching pairs $J \in \hat{\mathcal{B}}_{n-k+1}^{r-1}(U)$, $\omega \in \mathcal{B}_k^r(U)$, and $1 \leq r \leq \infty$.

It is worth considering continuity questions for this new version of the divergence theorem for open sets. For example, recall be the open set W in Figure 8 suppose $\{W_i\}$ is a sequence of open sets with smoothly embedded boundaries converging to W in the Hausdorff metric. Picture two lobes in W_i drawing closer together, and finally touching at their boundaries in the limit set W , as in the figure. Let $\omega_i \in \mathcal{B}_k^r(W_i)$ be a differential form which is identically

equal to a smooth differential form η_N in the top lobe, and another differential form η_S in the bottom lobe. We are used to seeing flux canceling in the limit if η_N and η_S extend to a smooth form for each i . But it is possible that $\omega_i \rightarrow \omega \in \mathcal{B}_k^r(W)$ and ω does not extend to a continuous form in $\mathcal{B}_k^r(\mathbb{R}^n)$. For example, $\omega_N = -y^2 dx/2 + x^2 dy/2$ and $\omega_S = y^2 dx/2 - x^2 dy/2$. Then $d\omega_N = (x+y)dxdy$ and $d\omega_S = -(x+y)dxdy$. So the net flux across the respective boundaries have opposite signs. In one lobe, the flux is outward flowing, and in the other, it is inward flowing. When we draw them together the flow moves from one lobe into the other and the velocity has an instantaneous change in direction and velocity as the flow crosses the frontier.

Corollary 6.9.2 (Kelvin-Stokes' theorem in open sets). *Let $0 \leq k \leq n-1$. Then*

$$\int_{\partial \perp J} \omega = \int_J \star d\omega$$

for all matching pairs $J \in \hat{\mathcal{B}}_{n-k-1}^{r-1}(U)$, $\omega \in \mathcal{B}_k^r(U)$, and $1 \leq r \leq \infty$.

7 Algebraic chains, submanifolds, soap films and fractals

7.1 Algebraic chains

An *algebraic k -cell* in an open set $U' \subseteq \mathbb{R}^n$ is a differential k -chain $F_*\tilde{Q}$ where Q is an affine k -cell contained in $U \subseteq \mathbb{R}^n$ and the map $F : U \rightarrow U'$ is an element of $\mathcal{M}^r(U, U')$. We say that $F_*\tilde{Q}$ is *non-degenerate* if $F : Q \rightarrow U'$ is a diffeomorphism onto its image. An *algebraic k -chain* A in U' is a finite sum of algebraic k -cells $A = \sum_{i=1}^N a_i F_{i*}\tilde{Q}_i$ where $a_i \in \mathbb{R}$, although it is often natural to assume $a_i \in \mathbb{Z}$ for an “integral” theory. According to Proposition 6.5.9 it follows that $\text{supp}(A) \subseteq U'$. Algebraic k -chains offer us relatively simple ways to represent familiar mathematical “objects” (see the figures below), and all of our integral theorems hold from the. For example, the change of variables equation for algebraic chains (see Corollary 6.5.8) takes the form

$$\int_{\sum_{i=1}^N a_i F_{i*}\tilde{Q}_i} \omega = \sum_{i=1}^N \int_{a_i Q_i} F_i^* \omega.$$

The integral on the right hand side is the Riemann integral for which there are classical methods of evaluation.

Singular cells are quite different from algebraic cells. A *singular cell* is defined to be a map of a closed cell $G : Q \rightarrow U$ and G might only be continuous. For example, let $G : [-1, 1] \rightarrow \mathbb{R}$ be given by $G(x) = x$ if $x \geq 0$ and $G(x) = -x$ if $x \leq 0$. This singular cell is nonzero, but the algebraic cell $G_*([0, 1]) = 0$. This problem of singular cells vanishes in homology, but the algebra inherent in algebraic chains is present before passing to homology.

7.1.1 Submanifolds of \mathbb{R}^n

We next show that smooth k -submanifolds in an open subset $U \subseteq \mathbb{R}^n$ are represented by algebraic k -chains.

Two non-degenerate algebraic k -cells $F_{1*}\tilde{Q}_1$ and $F_{2*}\tilde{Q}_2$ are *non-overlapping* if

$$\text{supp}(F_{1*}\tilde{Q}_1) \cap \text{supp}(F_{2*}\tilde{Q}_2) \subseteq \text{supp}(\partial F_{1*}\tilde{Q}_1) \cup \text{supp}(\partial F_{2*}\tilde{Q}_2)$$

That is, $F_1(Q_1)$ and $F_2(Q_2)$ intersect at most within their boundaries if they are non-overlapping. An algebraic chain $A = \sum F_{i*}\tilde{Q}_i$ is *non-overlapping* if each pair in the set $\{F_{i*}\tilde{Q}_i\}$ is non-overlapping. We say that algebraic k -chains A and A' are *equivalent* and write $A \sim A'$ if $A = A'$ as differential k -chains.

A k -chain $J \in \hat{\mathcal{B}}_k^r(U)$ represents a k -submanifold M of class $C^{r-1+Lip}$ in U if $\int_J \omega = \int_M \omega$ for all forms $\omega \in \mathcal{B}_k^r(U)$. If there exists a k -chain J representing M , then it is unique. Even though two algebraic chains A and A' may have different summands, they are still identical as chains if they both represent M .

Theorem 7.1.1. *Smoothly embedded compact k -submanifolds M of U of class $C^{r-1+Lip}$ are in one-to-one correspondence with equivalence classes of algebraic k -chains $A \in \hat{\mathcal{B}}_k^r(U)$ satisfying the following two properties:*

- (a) *For each $p \in \text{supp}(A)$ there exists a non-overlapping algebraic k -chain $A_p = \sum_{i=1}^N a_i F_{i*} \widetilde{Q}_i$ with $A_p \sim A$, $p \notin \cup_i \text{supp}(\partial F_{i*} \widetilde{Q}_i)$, and each F_{i*} is nondegenerate.*
- (b) *$\partial A = \{0\}$ for any A representing $[A]$.*

Furthermore, the correspondence is natural. That is, A corresponds to M if and only if A represents M .

Proof. Let M be a smoothly embedded compact k -submanifold of class $C^{r-1+Lip}$. Cover M with finitely many locally embedded charts diffeomorphic. The embeddings are of class B^r by Proposition 2.5.4. Let $p \in M$. Choose the cover C so that p does not meet any of the chart boundaries. Then M has a smooth triangulation with each closed simplex contained in an element of C . This can be easily proved by starting with any triangulation T of M , and then finding a subdivision T' such that the covering by closed simplices of T' refines the covering by the intersections with M of elements of C . We have $M = \cup_i F_i(\sigma^i)$ where σ^i is affine and F_i is a map of class B^r .

Then $A = \sum F_{i*} \widetilde{\sigma}^i$ is a non-overlapping algebraic k -chain. We show that A represents M . Suppose $\omega \in \mathcal{B}_k^r(U)$. Choose simplicial neighborhoods τ_i of σ^i such that F_i extends to an embedding of τ_i and $M = \cup_i F_i(\tau_i)$. Choose a partition of unity $\{\phi_i\}$ subordinate to $\{F_i \tau_i\}$. Using Theorem 2.9.4, Corollary 6.5.8, Corollary 6.6.2, and Corollary 4.2.2 we deduce

$$\begin{aligned} \int_M \omega &= \sum \int_{F_i \tau_i} \phi_i \omega = \int_{\tau_i} F_i^* \phi_i \omega = \int_{\widetilde{\tau}_i} F_i^* \phi_i \omega \\ &= \sum \int_{F_{i*} \widetilde{\tau}_i} \phi_i \omega = \sum \int_{m_{\phi_i} F_{i*} \widetilde{\tau}_i} \omega = \sum \int_{F_{i*} \widetilde{\sigma}^i} \omega = \int_A \omega. \end{aligned}$$

This shows that A satisfies condition (a) and represents M . Assuming M has no boundary, it follows that $\partial A = 0$.

Conversely, let $A = \sum_i F_{i*} \widetilde{Q}_i$ be a non-overlapping algebraic k -chain satisfying (a) and (b). We show that $M = \text{supp}(A)$ is a smoothly embedded compact k -submanifold. For each $p \in M$ we may choose a non-overlapping representative $A_p = \sum G_{i*} \widetilde{Q}_i'$ of the equivalence class $[A]$ such that $p \notin \text{supp}(\partial G_{i*} \widetilde{Q}_i')$ for any i by (a). Then there exists Q_{i_p} such that $p \in G_{i_p}(Q_{i_p})$. We show that $\{Q_{i_p}\}$ determines an atlas of M : If $x \in Q_{i_p} \cap Q_{i_q}$, then $Q_{i_x} \cap Q_{i_p} \cap Q_{i_q}$ is smoothly embedded. It follows that the overlap maps are of class B^r since all of the embeddings are of class B^r . Therefore, by Proposition 2.5.4, M is a smoothly embedded submanifold of U of class $C^{r-1+Lip}$. \square

If we replace condition (b) with ∂A is nonzero and represents a smooth $(k-1)$ -submanifold with smooth boundary, then A represents a k -submanifold with smooth boundary. If $A = \sum A_j$, and each A_j represents an embedded submanifold with smooth boundary, then A represents a piecewise smooth immersed submanifold.

Examples 7.1.2.

- *The differential chain representative of an oriented 2-sphere in \mathbb{R}^3 can be written as the sum of representatives of any two hemispheres, for example, or as the sum of representatives of puzzle pieces as seen in wikipedia's symbol.*

- In classical topology a torus can be obtained by gluing a cylinder to the sphere with two small disks removed. Instead, we subtract representatives of the small disks from a representative of the sphere, and add a representative of the cylinder to get a representative of the torus. Similarly, any orientable surface may be represented by a chain by adding representatives of “handles” to a sphere representative. As with the Cantor set or Sierpinski triangle below, algebraic sums and pushforward can replace cutting and pasting.
- The quadrifolium and Boys surface can be represented by algebraic chains.

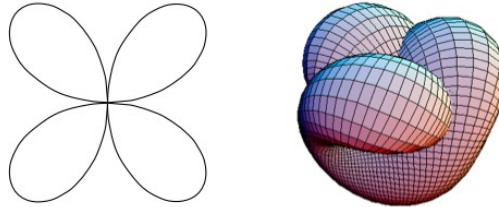


Figure 10

- Whitney stratified sets can be represented by algebraic chains since stratified sets can be triangulated [Gor78] (see Figure 11).

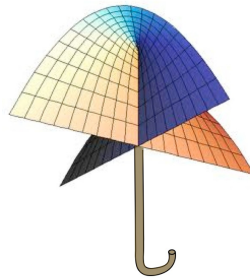


Figure 11: Whitney umbrella

7.1.2 Representatives of fractals

The interior of the Sierpinski triangle T can be represented by a 2-chain $\tilde{T} = \lim_{k \rightarrow \infty} (4/3)^k \tilde{S}_{k_i}$ in the B^1 norm where the S_{k_i} are oriented simplices filling up the interior of T in the standard construction²². The support of its boundary $\partial\tilde{T}$ is T . We may integrate smooth forms and apply the integral theorems to calculate flux, etc. Other applications to fractals may be found in [Har99, Har98].

We revisit the middle third Cantor set $\Gamma \subset I$, the unit interval from §2.10.1. Recall the chain representatives of approximating open sets obtained by removing middle thirds forms a Cauchy sequence in $\hat{\mathcal{B}}_1^1(I)$. These are algebraic 1-chains. The limit $\tilde{\Gamma}$ is a differential 1-chain that represents Γ . Its boundary $\partial\tilde{\Gamma}$ is well-defined and is supported in

²²If we had started with polyhedral chains instead of Dirac chains, convergence would be in the B^0 norm.

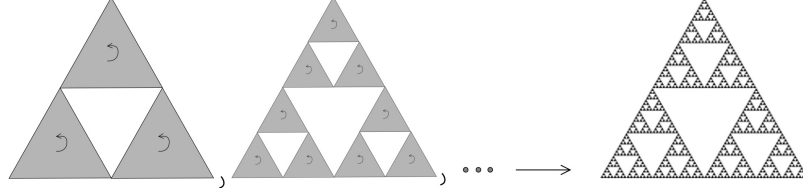


Figure 12: Sierpinski triangle

the classical middle third Cantor set. We may therefore integrate differential forms over Γ and state the fundamental theorem of calculus where Γ is a domain and $f \in \mathcal{B}_1^1(I)$:

$$\int_{\partial\Gamma} f = \int_{\Gamma} df.$$

8 Vector fields and the primitive operators in open sets

8.1 The space $\mathcal{V}^r(U)$ of vector fields

Let U be open in \mathbb{R}^n and $V : U \rightarrow \mathbb{R}^n$ a vector field on U . Recall V^b is the differential 1-form associated to V by way of the inner product²³. The Banach space of vector fields V with $\|V^b\|_{B^r} < \infty$ is denoted $\mathcal{V}^r(U)$. The norms are increasing with r . It follows that the projective limit $\mathcal{V}(U) = \mathcal{V}^\infty(U) := \varprojlim \mathcal{V}^r(U)$, endowed with the projective limit topology, is a Fréchet space.

Lemma 8.1.1. *Let $U \subset \mathbb{R}^n$ be open and $V : U \rightarrow \mathbb{R}^n$ a vector field. Then $V \in \mathcal{V}^r(U)$ if and only if each coordinate function f_i satisfies $\|f_i\|_{B^{r,U}} \leq \|V^b\|_{B^{r,U}}$ for all $0 \leq r < \infty$.*

Proof. The proof is immediate from the definitions. □

We next extend the primitive operators to E_V, E_V^\dagger , and P_V where $V \in \mathcal{V}^r(U), r \geq 1$ and U is an open subset of \mathbb{R}^n , as well as multiplication by a function m_f . However, we found it advantageous to work first with constant vector fields to define the boundary operator and establish its basic properties.

8.2 Extrusion

Definition 8.2.1. *Define the bilinear map*

$$\begin{aligned} E : \mathcal{V}^r(U) \times \mathcal{A}_k(U) &\rightarrow \mathcal{A}_{k+1}(U) \\ (V, (p; \alpha)) &\mapsto (p; V(p) \wedge \alpha). \end{aligned}$$

Let $E_V(p; \alpha) := E(V, (p; \alpha))$.

²³The choice of inner product on \mathbb{R}^n has no significant influence on the theory. The spaces are independent of the choice, as the norms are comparable.

It follows that $i_V \omega(p; \alpha) = \omega E_V(p; \alpha)$ where i_V is classical interior product.

Theorem 8.2.2. *Let $0 \leq k \leq n-1$ and $r \geq 1$. The bilinear map $E = E_k^r : \mathcal{V}^r(U) \times \mathcal{A}_k(U) \rightarrow \mathcal{A}_{k+1}(U)$ extends to a separately continuous bilinear map $E = E_k^r : \mathcal{V}^r(U) \times \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k+1}^r(U)$ with*

$$\|E(V, J)\|_{B^r, U} \leq n^2 2^r \|V^b\|_{B^r, U} \|J\|_{B^r, U} \text{ for all } J \in \hat{\mathcal{B}}_k^r(U).$$

Furthermore, for each $0 \leq k \leq n-1$, there exists a separately continuous bilinear map $E = E_k : \mathcal{V}^\infty(U) \times \hat{\mathcal{B}}_k(U) \rightarrow \hat{\mathcal{B}}_k(U)$ which restricts to E_k^r on each $\nu_k^r(\mathcal{V}^r(U)) \times u_k^r(\hat{\mathcal{B}}_k^r(U))$.

Proof. We establish the inequality. Since $V \in \mathcal{V}^r(U)$ we know $V = \sum f_i e_i$ where $f_i \in \mathcal{B}_0^r(U)$. Lemma 4.1.3 readily extends to $f_i \in \mathcal{B}_0^r(U)$, and thus $E_V = \sum_{i=1}^n E_{f_i e_i} = \sum_{i=1}^n m_{f_i} E_{e_i}$.

By Theorem 6.6.1 $\|m_{f_i} A\|_{B^r, U} \leq n 2^r \|f_i\|_{B^r, U} \|A\|_{B^r, U}$. Therefore, using Corollary 3.1.5 and Lemma 8.1.1,

$$\begin{aligned} \|E_V A\|_{B^r, U} &\leq \sum_{i=1}^n \|m_{f_i} E_{e_i} A\|_{B^r, U} \leq n 2^r \sum_{i=1}^n \|f_i\|_{B^r, U} \|E_{e_i} A\|_{B^r, U} \leq n 2^r \sum_{i=1}^n \|f_i\|_{B^r, U} \|A\|_{B^r, U} \\ &\leq n^2 2^r \|V^b\|_{B^r, U} \|A\|_{B^r, U}. \end{aligned}$$

Define $E_V(J) := \lim_{i \rightarrow \infty} E_V(A_i)$ for $J \in \hat{\mathcal{B}}(U)$. It follows that $\|E(V, J)\|_{B^r, U} \leq n^2 r \|V^b\|_{B^r, U} \|J\|_{B^r, U}$.

For the last assertion, first fix $V \in \mathcal{V}^\infty(U)$. Since $E_V(H_k(U) \subset H_{k+1}(U))$, we can apply Theorem 6.1.5 to extend $E_V : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k+1}^r(U)$ to a continuous linear map $E_V : \hat{\mathcal{B}}_k(U) \rightarrow \hat{\mathcal{B}}_{k+1}(U)$. Define $E : \mathcal{V}^\infty(U) \times \hat{\mathcal{B}}_k(U) \rightarrow \hat{\mathcal{B}}_k(U)$ by $E(V, J) := E_V(J)$. This is well-defined since $V \in \mathcal{V}^\infty(U)$ implies $V \in \mathcal{V}^r(U)$. Last of all, we establish continuity in the first variable: Suppose $J \in \hat{\mathcal{B}}_k(U)$. Then there exists $r \geq 0$ and $J^r \in \hat{\mathcal{B}}_k^r(U)$ such that $u_k^r(J^r) = J$. Suppose $V_i \rightarrow 0$ in $\mathcal{V}^\infty(U)$. Since the inclusion $\mathcal{V}^\infty(U) \rightarrow \mathcal{V}^{r+1}(U)$ is the identity map and continuous, then $V_i \rightarrow 0$ in $\mathcal{V}^r(U)$. Now use the fact that $E_k^r : \mathcal{V}^{r+1}(U) \times \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k+1}^r(U)$ is continuous in the first variable. □

The next result establishes duality of extrusion E_V with the classical interior product of differential forms i_V .

Theorem 8.2.3 (Change of dimension I). *Let $0 \leq k \leq n-1$. Then*

$$\int_{E_V J} \omega = \int_J i_V \omega \tag{8.1}$$

for all matching triples $V \in \mathcal{V}^r(U)$, $J \in \hat{\mathcal{B}}_k^r(U)$, $\omega \in \mathcal{B}_{k+1}^r(U)$, and $1 \leq r \leq \infty$.

Proof. Since $\omega(E_V(p; \alpha)) = i_V \omega(p; \alpha)$ the integral relation holds for Dirac chains, and thus for all matching pairs of chains and forms by separate continuity of the integral pairing. □

8.3 Retraction

Definition 8.3.1. *Define the bilinear map*

$$\begin{aligned} E^\dagger : \mathcal{V}^r(U) \times \mathcal{A}_{k+1}(U) &\rightarrow \mathcal{A}_k(U) \\ (p; v_1 \wedge \cdots \wedge v_{k+1}) &\mapsto \sum_{i=1}^{k+1} (-1)^{i+1} \langle V(p), v_i \rangle (p; v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_{k+1}). \end{aligned}$$

Let $E_V^\dagger(p; \alpha) := E^\dagger(V, (p; \alpha))$, or, in our more compact notation,

$$E_V^\dagger(p; \alpha) = \sum_{i=1}^{k+1} (-1)^{i+1} \langle V(p), v_i \rangle (p; \hat{\alpha}_i).$$

Lemma 8.3.2. *If $v \in \mathbb{R}^n$ and $f \in \mathcal{B}_0^r(U)$, $r \geq 1$, then $E_{fv}^\dagger = m_f E_v^\dagger$.*

Proof. This follows directly from the definitions:

$$E_{fv}^\dagger(p; \alpha) = \sum_{i=1}^{k+1} (-1)^{i+1} \langle f(p)v, v_i \rangle (p; \hat{\alpha}_i) = f(p) \sum_{i=1}^{k+1} (-1)^{i+1} \langle v, v_i \rangle (p; \hat{\alpha}_i) = m_f E_v^\dagger(p; \alpha).$$

□

Theorem 8.3.3. *Let $1 \leq k \leq n$ and $r \geq 1$. The bilinear map determined by*

$$\begin{aligned} E^\dagger &= (E^\dagger)_k^r : \mathcal{V}^r(U) \times \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k-1}^r(\mathbb{R}^n)(U) \\ (V, J) &\mapsto E_V^\dagger(J) \end{aligned}$$

is well-defined and separately continuous with $\|E^\dagger(V, J)\|_{B^{r,U}} \leq k \binom{n}{k} \|V^\flat\|_{B^{r,U}} \|J\|_{B^{r,U}}$. Furthermore, for each $1 \leq k \leq n$, there exists a continuous bilinear map $E^\dagger = E_k^\dagger : \mathcal{V}^\infty(U) \times \hat{\mathcal{B}}_k(U) \rightarrow \hat{\mathcal{B}}_{k-1}(U)$ which restricts to $(E^\dagger)_k^r$ on each $u_k^r(\hat{\mathcal{B}}_k^r(U))$.

The proof is similar to that of Theorem 3.2.2 and uses $E_V^\dagger(H_k(U) \subset H_k(U))$.

Lemma 8.3.4. *Let $V \in \mathcal{V}^r(U)$ be a vector field. Then $V^\flat \wedge (\cdot) : \mathcal{B}_k^r(U) \rightarrow \mathcal{B}_{k+1}^r(U)$ is continuous and $(V^\flat \wedge \omega)(p; \alpha) = \omega E_V^\dagger(p; \alpha)$.*

Proof. If $V \in \mathcal{V}^r(U)$, then $v^\flat \wedge \omega \in \mathcal{B}_{k+1}^r(U)$ for all $\omega \in \mathcal{B}_k^r(U)$ and $V^\flat \in \mathcal{B}_1^r(U)$. Finally,

$$\omega E_V^\dagger(p; \alpha) = \sum_{i=1}^k (-1)^{i+1} \langle V(p), v_i \rangle \omega(p; \hat{\alpha}_i) = (V^\flat \wedge \omega)(p; \alpha) \quad (8.2)$$

by the standard formula of wedge product of forms (see [Fed69] §1.4, for example.) □

Theorem 8.3.5 (Change of dimension II). *Let $1 \leq k \leq n$. Then*

$$\int_{E_V^\dagger J} \omega = \int_J V^\flat \wedge \omega \quad (8.3)$$

for all matching triples $V \in \mathcal{V}^r(U)$, $J \in \hat{\mathcal{B}}_k^r(U)$, $\omega \in \mathcal{B}_{k-1}^r(U)$, and $1 \leq r \leq \infty$.

The proof is similar to that of Corollary 3.2.5.

8.4 Prederivative

Prederivative gives us a way to “geometrically differentiate” a differential chain in the infinitesimal directions determined by a vector field, even when the support of the differential chain is highly nonsmooth, and without reference to any functions or differential forms.

Definition 8.4.1. Define the bilinear map $P : \mathcal{V}^{r+1}(U) \times \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k^{r+1}(U)$ by $P(V, J) = P_V(J) := \partial E_V(J) + E_V \partial(J)$.

Since both E_V and ∂ are continuous, then P_V is continuous.

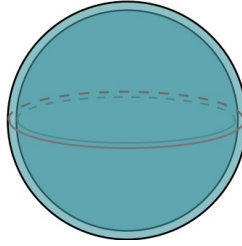


Figure 13: Prederivative of the sphere with respect to the radial vector field

Its dual operator L_V is the classically defined Lie derivative since the relation $L_V = i_V d + di_V$ uniquely determines it.

Theorem 8.4.2. Let $0 \leq k \leq n$ and $r \geq 1$. The bilinear map determined by

$$P = P_k^r : \mathcal{V}^{r+1}(U) \times \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k^{r+1}(U) \\ (V, J) \mapsto P_V(J)$$

is well-defined and separately continuous with $\|P(V, J)\|_{B^{r+1, U}} \leq 2kn^3 2^r \|V^\flat\|_{B^{r+1, U}} \|J\|_{B^{r, U}}$. Furthermore, for each $0 \leq k \leq n$, there exists a separately continuous bilinear map $P = P_k : \mathcal{V}^\infty(U) \times \hat{\mathcal{B}}_k(U) \rightarrow \hat{\mathcal{B}}_k(U)$ which restricts to P_k^r on each $u_k^r(\hat{\mathcal{B}}_k^r(U))$.

Proof. By Definition 8.4.1, and Theorems 3.5.1, and 8.2.2

$$\begin{aligned} \|P_V J\|_{B^{r+1, U}} &\leq \|\partial E_V J\|_{B^{r+1, U}} + \|E_V \partial J\|_{B^{r+1, U}} \leq kn \|E_V J\|_{B^{r, U}} + n^2 2^r \|V^\flat\|_{B^{r+1, U}} \|\partial J\|_{B^{r+1, U}} \\ &\leq kn^3 2^r \|V^\flat\|_{B^{r, U}} \|J\|_{B^{r, U}} + k(n-1)n^2 2^r \|V^\flat\|_{B^{r+1, U}} \|J\|_{B^{r, U}} \\ &\leq 2kn^3 2^r \|V^\flat\|_{B^{r+1, U}} \|J\|_{B^{r, U}}. \end{aligned}$$

For the last assertion, we first establish continuity of P in the first variable: Suppose $J \in \hat{\mathcal{B}}_k(U)$. Then there exists $r \geq 0$ and $J^r \in \hat{\mathcal{B}}_k^r(U)$ such that $u_k^r(J^r) = J$. Suppose $V_i \rightarrow 0$ in $\mathcal{V}^\infty(U)$. Since the inclusion $\mathcal{V}^\infty(U) \rightarrow \mathcal{V}^{r+1}(U)$ is the identity map and continuous, then $V_i \rightarrow 0$ in $\mathcal{V}^{r+1}(U)$. Now use the fact that $P_k^r : \mathcal{V}^{r+1}(U) \times \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k^{r+1}(U)$ is continuous in the first variable. Continuity in the second variable of P follows from Theorem 6.1.5 since $P_V(H_k(U) \subset H_k(U))$. \square

Theorem 8.4.3 (Change of order). *Let $0 \leq k \leq n$. Then*

$$\int_{P_V J} \omega = \int_J L_V \omega \quad (8.4)$$

for all matching triples $V \in \mathcal{V}^{r+1}(U)$, $J \in \hat{\mathcal{B}}_k^r(U)$, $\omega \in \mathcal{B}_k^{r+1}(U)$, and $1 \leq r \leq \infty$.

Theorem 8.4.4. *If $V \in \mathcal{V}^{r+1}(U)$ is a vector field and $J \in \hat{\mathcal{B}}_k^r(U)$ has compact support $\text{supp}(J) \subset U$, then*

$$P_V J = \lim_{t \rightarrow 0} (\phi_{t*} - \text{Id})/t(J)$$

where ϕ_t is the time- t map of the flow of V .

Proof. Since $\text{supp}(J) \subset U$ there exists an open set $\text{supp}(J) \subset U' \subset U$. Since $\text{supp}(J)$ is compact, there exists $t_0 > 0$ such that $\phi_t(p)$ is defined for all $0 \leq t \leq t_0$ and $p \in \text{supp}(J)$. If $\omega \in \mathcal{B}_k^{r+1}(U)$, then $L_V \omega = \lim_{t \rightarrow 0} ((\phi_t^* - \text{Id})/t(\omega)) \in \mathcal{B}_k^r(U)$. Let $F_t = \phi_t - \text{Id}$. Then $\|L_V \omega - F_t^*(\omega)\|_{B^r, U} \rightarrow 0$ as $t \rightarrow 0$.

There exists $\eta \in \mathcal{B}_k^{r+1}(U)$ with $\|\eta\|_{B^{r+1}, U} = 1$ and $\|(F_{t*} - P_V)J\|_{B^{r+1}, U} = \int_{(F_{t*} - P_V)J} \eta = |\int_J (F_t^* - L_V)\eta| \leq \|J\|_{B^r, U'} \|(F_t^* - L_V)\eta\|_{B^r, U'} \rightarrow 0$ as $t \rightarrow 0$. Hence $P_V J = \lim_{t \rightarrow 0} F_{t*} J = \lim_{t \rightarrow 0} (\phi_{t*} - \text{Id}_*)(J/t)$. \square

The next result follows directly from the definitions.

Theorem 8.4.5 (Commutation Relations). *If $V_1, V_2 \in \mathcal{V}^r(U)$, then $[P_{V_1}, P_{V_2}] = P_{[V_1, V_2]}$, $\{E_{V_1}^\dagger, E_{V_2}^\dagger\} = \{E_{V_1}, E_{V_2}\} = 0$, $\{E_{V_1}, E_{V_2}^\dagger\} = \langle V_1(p), V_2(p) \rangle I$, $[E_{V_2}^\dagger, P_{V_1}] = E_{[V_1, V_2]}^\dagger$, and $[E_{V_2}, P_{V_1}] = E_{[V_1, V_2]}$.*

Let $\mathcal{L}(\hat{\mathcal{B}}(U))$ be the algebra of operators on $\hat{\mathcal{B}}(U)$. This includes $\{m_f, F_*, \partial, \perp\}$, as well as E_V, E_V^\dagger, P_V , for vector fields $V \in \mathcal{V}^r(U)$. These operators and their corresponding integral theorems extend immediately to $\hat{\mathcal{B}}(U)$. We obtain the algebra of operators $\mathcal{L}(\hat{\mathcal{B}}(U))$.

8.4.1 Dipole chains

$P_v \tilde{\sigma}$ is a *dipole k -cell*, and $\sum F_{*i} P_{v_i} \tilde{s}_i$ is a *dipole algebraic k -chain*, or, more simply, a *dipole k -chain*. This idea may be extended to define k -cells of order r , but we do not do so here. Dipole k -chains are useful for representing soap films, Moebius strips, and soap bubbles. (See [Har04a] and [Har12d] for more details).

8.5 Naturality of the operators

Theorem 8.5.1 (Naturality of the operators). *Suppose $V \in \mathcal{V}^r(U)$ is a vector field and $F \in \mathcal{M}^r(U, U')$ is a map. Then*

(a) $F_* m_{f \circ F} = m_f F_*$ for all $f \in \mathcal{B}_0^r(U')$ for $1 \leq r \leq \infty$.
If F is a diffeomorphism onto its image, then

(b) $F_* E_V = E_{FV} F_*$;

(c) $F_* E_V^\dagger = E_{FV}^\dagger F_*$ if F preserves the metric;

$$(d) F_* P_V = P_{FV} F_*.$$

Proof. (a): $F_* m_{f \circ F}(p; \alpha) = (F(p); f(F(p)) F_* \alpha) = m_f F_*(p; \alpha)$. (b): is omitted since it is much like (c). (c): Since $\langle F_* V, F_* v_i \rangle = \langle V, v_i \rangle$ we have

$$\begin{aligned} F_*(E_V^\dagger((p; \alpha))) &= \sum_{i=1}^k (-1)^{i+1} \langle V, v_i \rangle \langle F(p); F_* \hat{\alpha}_i \rangle = \sum_{i=1}^k (-1)^{i+1} \langle F_* V, F_* v_i \rangle \langle F(p); F_* \hat{\alpha}_i \rangle \\ &= E_{FV}^\dagger(F(p); F_* \alpha) \\ &= E_{FV}^\dagger F_*(p; \alpha) \end{aligned}$$

(d): Observe that if V_t is the flow of V , then $FV_t F^{-1}$ is the flow of FV . It follows that

$$\begin{aligned} F_* P_v(p; \alpha) &= F_* \lim_{t \rightarrow 0} (V_t(p); V_{t*} \alpha / t) - (p; \alpha / t) = \lim_{t \rightarrow 0} (F(V_t(p)); F_* V_{t*} \alpha / t) - (F(p); F_* \alpha / t) \\ &= \lim_{t \rightarrow 0} ((FV_t F^{-1}) F(p); F_* \alpha / t) - (F(p); F_* \alpha / t) \\ &= P_{FV}(F(p); F_* \alpha) \\ &= P_{FV} F_*(p; \alpha). \end{aligned}$$

□

Corollary 8.5.2. $[F_*, \partial] = 0$ for maps $F \in \mathcal{M}^r(U, U')$ and $r \geq 1$.

Proof. This follows from Theorem 8.5.1 and the definition of ∂ . □

Our results regarding the operators $m_f \in \mathcal{L}(\hat{\mathcal{B}}(\mathbb{R}^n))$ extend to similar results for $m_f \in \mathcal{L}(\hat{\mathcal{B}}(U))$.

8.5.1 Representatives of vector fields

We say that a k -chain $\tilde{X} \in \hat{\mathcal{B}}_k^r(U)$, $r \geq 1$ represents a k -vector field $X : U \rightarrow \Lambda_k$ if

$$\int_{\tilde{X}} \omega = \int_U \omega(X(p)) dV$$

for all $\omega \in \mathcal{B}_k^r(U)$.

Theorem 8.5.3. If $X \in \mathcal{V}^r(U)$ is a k -vector field where U is a bounded and open set for $r \geq 1$, then there exists a differential k -chain $\tilde{X} \in \hat{\mathcal{B}}_k^r(U)$ which represents X .

Proof. Let $\tilde{U} \in \hat{\mathcal{B}}_n^1(U)$ the n -chain representing U (see §2.10.3). Given $\alpha \in \Lambda_k$ then $E_\alpha \perp \tilde{U} \in \hat{\mathcal{B}}_k^1(U)$ represents the constant k -vector field $(p; \alpha)$ defined over U since for all $\omega \in \mathcal{B}_k^1(U)$, we have

$$\int_{E_\alpha \perp \tilde{U}} \omega = \int_{\perp \tilde{U}} i_\alpha \omega = \int_{\tilde{U}} \star i_\alpha \omega = \int_U \star i_\alpha \omega = \int_U \omega(p; \alpha) dV$$

using Corollaries 3.1.5, 3.6.3, Theorem 2.9.4, and the definition of interior product. Since the norms decrease, it follows that $E_\alpha \perp \tilde{U} \in \hat{\mathcal{B}}_k^r(U)$ also represents the constant k -vector field $(p; \alpha)$, but using test forms in $\mathcal{B}_k^r(U)$.

Now suppose $X(p) = \sum_I (p; f_I(p) \alpha_I)$ is a k -vector field in U where $f_I \in \mathcal{B}_0^r(U)$. Then $\tilde{X} = \sum_I m_{f_I} E_{\alpha_I} \perp \tilde{U} \in \hat{\mathcal{B}}_k^r(U)$ represents X since

$$\int_{m_{f_I} E_{\alpha_I} \perp \tilde{U}} \omega = \int_{E_{\alpha_I} \perp \tilde{U}} f_I \omega = \int_U f_I \omega(p; \alpha_I) dV.$$

Therefore,

$$\int_{\tilde{X}} \omega = \int_U \left(\sum_I f_I \omega \right) (p; \alpha_I) dV = \int_U \omega \left(\sum_I (p; f_I(p) \alpha_I) \right) dV = \int_U \omega(X(p)) dV.$$

□

For $k = 0$, we have found a 0-chain \tilde{f} representing a function $f \in \mathcal{B}_0^1(U)$. That is,

$$\int_{\tilde{f}} g = \int_U f \cdot g dV = \int_U f \wedge \star g$$

for all $g \in \mathcal{B}_0^1(U)$. As an example of this construction, we have the following corollary:

Corollary 8.5.4. *Suppose $f : U \rightarrow \mathbb{R}$ is a Lipschitz function where U is bounded and open in \mathbb{R}^n . Then*

$$\int_{\perp \partial \perp \tilde{f}} \omega = - \int_U df \wedge \star \omega$$

for all $\omega \in \mathcal{B}_1^2(U)$.

Proof. Let $\omega \in \mathcal{B}_1^2(U)$. By Theorem 3.6.3 and Stokes' Theorem 3.5.4 we have $\int_{\perp \partial \perp \tilde{f}} \omega = \int_{\tilde{f}} \star d \star \omega = \int_U f \wedge d \star \omega = - \int_U df \wedge \star \omega$. □

In closing, we remark that the definitions and results of this section readily extend to k -vector fields.

9 Cartesian wedge product

9.1 Definition of $\hat{\times}$

Suppose $U_1 \subseteq \mathbb{R}^n$ and $U_2 \subseteq \mathbb{R}^m$ are open and contain the respective origins. Let $\iota_1 : U_1 \rightarrow U_1 \times U_2$ and $\iota_2 : U_2 \rightarrow U_1 \times U_2$ be the inclusions $\iota_1(p) = (p, 0)$ and $\iota_2(q) = (0, q)$. Let $\pi_1 : U_1 \otimes U_2 \rightarrow U_1$ and $\pi_2 : U_1 \otimes U_2 \rightarrow U_2$ be the projections $\pi_i(p_1, p_2) = p_i$, $i = 1, 2$. Let $(p; \alpha) \in \mathcal{A}_k(U_1)$ and $(q; \beta) \in \mathcal{A}_\ell(U_2)$.

Definition 9.1.1. *Define $\hat{\times} : \mathcal{A}_k(U_1) \times \mathcal{A}_\ell(U_2) \rightarrow \mathcal{A}_{k+\ell}(U_1 \times U_2)$ by $\hat{\times}((p; \alpha), (q; \beta)) := ((p, q); \iota_{1*} \alpha \wedge \iota_{2*} \beta)$ where $(p; \alpha)$ and $(q; \beta)$ are k - and ℓ -elements, respectively, and extend bilinearly.*

We call $P \hat{\times} Q = \hat{\times}(P, Q)$ the *Cartesian wedge product*²⁴ of P and Q . Cartesian wedge product of Dirac chains is associative since wedge product is associative, but it is not graded commutative since Cartesian product is not graded commutative. We next show that Cartesian wedge product is continuous.

²⁴By the universal property of tensor product, $\hat{\times}$ factors through a continuous linear map *cross product* $\tilde{\times} : \hat{\mathcal{B}}_j(U_1) \otimes \hat{\mathcal{B}}_k(U_2) \rightarrow \hat{\mathcal{B}}_{j+k}(U_1 \times U_2)$. This is closely related to the classical definition of *cross product* on simplicial chains ([Hat01], p. 278)

Lemma 9.1.2. *If D^i is an i -difference k -chain in U_1 and E^j is a j -difference ℓ -chain in U_2 , then $D^i \hat{\times} E^j$ is an $(i+j)$ -difference $(k+\ell)$ -chain in $U_1 \times U_2$ with*

$$\|D^i \hat{\times} E^j\|_{B^{i+j}, U_1 \times U_2} \leq |D^i|_{B^i, U_1} |E^j|_{B^j, U_2}.$$

Proof. This follows since $\Delta_{\sigma^i}(p; \alpha) \hat{\times} \Delta_{\tau^j}(q; \beta) = \Delta_{\sigma^i, \tau^j}((p, q); \alpha \wedge \beta)$ and $|\Delta_{\sigma^i, \tau^j}((p, q); \alpha \wedge \beta)|_{i+j, U_1 \times U_2} \leq \|\sigma\| \|\tau\| \|\alpha\| \|\beta\| = |\Delta_{\sigma^i}(p; \alpha)|_{B^i, U_1} |\Delta_{\tau^j}(q; \beta)|_{B^j, U_2}$. Furthermore, if $\Delta_{\sigma^i}(p; \alpha)$ is in U_1 and $\Delta_{\tau^j}(q; \beta)$ is in U_2 , then the paths $\ell(p, \sigma)$ (see §6.1) are contained in U_1 and the paths $\ell(q, \tau)$ are contained in U_2 . It follows from the definition of Cartesian product that the paths $\ell((p, q), \sigma \circ \tau)$ are contained in $U_1 \times U_2$. Therefore, $D^i \hat{\times} E^j$ is an $(i+j)$ -difference $(k+\ell)$ -chain in $U_1 \times U_2$. \square

Proposition 9.1.3. *Suppose $P \in \mathcal{A}_k(U_1)$ and $Q \in \mathcal{A}_\ell(U_2)$ are Dirac chains where $U_1 \subseteq \mathbb{R}^n, U_2 \subseteq \mathbb{R}^m$ are open. Then $P \hat{\times} Q \in \mathcal{A}_{k+\ell}(U_1 \times U_2)$ with*

$$\|P \hat{\times} Q\|_{B^{r+s}, U_1 \times U_2} \leq \|P\|_{B^r, U_1} \|Q\|_{B^s, U_2}.$$

Furthermore,

$$\|P \hat{\times} \widetilde{(a, b)}\|_{B^r, U_1 \times \mathbb{R}} \leq |b - a| \|P\|_{B^r, U_1}$$

where $\widetilde{(a, b)}$ is the 1-chain representing the interval (a, b) .

Proof. Choose $\epsilon > 0$ and let $\epsilon' = \epsilon / (\|P\|_{B^r, U_1} + \|Q\|_{B^s, U_2} + 1) < 1$. There exist decompositions $P = \sum_{i=0}^r D^i$ and $Q = \sum_{j=0}^s E^j$ such that $\|P\|_{B^r} > \sum_{i=0}^r |D^i|_{B^i, U_1} - \epsilon$ and $\|Q\|_{B^s, U_2} > \sum_{j=0}^s |E^j|_{B^j, U_2} - \epsilon$. Since $P \hat{\times} Q = \sum_{i=0}^r \sum_{j=0}^s D^i \hat{\times} E^j$ and by Lemma 9.1.2

$$\begin{aligned} \|P \hat{\times} Q\|_{B^{r+s}, U_1 \times U_2} &\leq \sum_{i=0}^r \sum_{j=0}^s \|D^i \hat{\times} E^j\|_{B^{i+j}, U_1 \times U_2} \\ &\leq \sum_{i=0}^r \sum_{j=0}^s |D^i|_{B^i, U_1} |E^j|_{B^j, U_2} \\ &\leq (\|P\|_{B^r, U_1} + \epsilon') (\|Q\|_{B^s, U_2} + \epsilon'). \end{aligned}$$

Since this holds for all $\epsilon > 0$, the result follows.

The second inequality is similar, except we use $\|D^i \hat{\times} \widetilde{(a, b)}\|_{B^i, U_1 \times \mathbb{R}} \leq |b - a| |D^i|_{B^i, U_1}$. \square

Let $J \in \hat{\mathcal{B}}_k^r(U_1)$ and $K \in \hat{\mathcal{B}}_\ell^s(U_2)$. Choose Dirac chains $P_i \rightarrow J$ converging in $\hat{\mathcal{B}}_k^r(U_1)$, and $Q_i \rightarrow K$ converging in $\hat{\mathcal{B}}_\ell^s(U_2)$. Proposition 9.1.3 implies that $\{P_i \hat{\times} Q_i\}$ is Cauchy.

Definition 9.1.4. *Define*

$$J \hat{\times} K := \lim_{i \rightarrow \infty} P_i \hat{\times} Q_i.$$

Theorem 9.1.5. *Cartesian wedge product $\hat{\times} : \hat{\mathcal{B}}_k^r(U_1) \times \hat{\mathcal{B}}_\ell^s(U_2) \rightarrow \hat{\mathcal{B}}_{k+\ell}^{r+s}(U_1 \times U_2)$ is associative, bilinear and continuous for all open sets $U_1 \subseteq \mathbb{R}^n, U_2 \subseteq \mathbb{R}^m$ and satisfies*

$$(a) \quad \|J \hat{\times} K\|_{B^{r+s}, U_1 \times U_2} \leq \|J\|_{B^r, U_1} \|K\|_{B^s, U_2}$$

(b) $\|J \hat{\times} \widetilde{(a, b)}\|_{B^r, U_1 \times \mathbb{R}} \leq |b - a| \|J\|_{B^r, U_1}$ where $(a, b) \subset \mathbb{R}$.

(c)

$$\partial(J \hat{\times} K) = \begin{cases} (\partial J) \hat{\times} K + (-1)^k J \hat{\times} (\partial K), & k > 0, \ell > 0 \\ (\partial J) \hat{\times} K, & k > 0, \ell = 0 \\ J \hat{\times} (\partial K), & k = 0, \ell > 0 \end{cases}$$

(d) $J \hat{\times} K = 0$ implies $J = 0$ or $K = 0$;

(e) $(p; \sigma \otimes \alpha) \hat{\times} (q; \tau \otimes \beta) = ((p, q); \sigma \otimes \tau \otimes \iota_{1*} \alpha \wedge \iota_{2*} \beta)$ where $\sigma \otimes \alpha$ and $\tau \otimes \beta$ are elements of the Koszul complex $\bigoplus_{s=0}^{\infty} \bigoplus_{k=0}^n S^s \otimes \Lambda_k$ (see Remark 3.4.2);

(f) $(\pi_{1*} \omega \wedge \pi_{2*} \eta)(J \hat{\times} K) = \omega(J) \eta(K)$ for $\omega \in \mathcal{B}_k^r(U_1), \eta \in \mathcal{B}_\ell^s(U_2)$;

(g) $\text{supp}(J \hat{\times} K) = \text{supp}(J) \times \text{supp}(K)$.

Proof. (a) and (b): These are a consequence of Proposition 9.1.3.

(c): This follows from Theorem 3.5.1: $\partial((p, q); \alpha \wedge \beta) = (\partial((p, q); \alpha)) \cdot ((p, q); \beta) + (-1)^k ((p, q); \alpha) \cdot (\partial((p, q); \beta)) = (\partial(p; \alpha)) \hat{\times} (q; \beta) + (-1)^k (p; \alpha) \hat{\times} (\partial(q; \beta))$, for all $(p; \alpha) \in \mathcal{A}_k(U_1)$, and $(q; \beta) \in \mathcal{P}_\ell(U_2)$. The boundary relations for Dirac chains follow by linearity. We know $\hat{\times}$ is continuous by (a) and ∂ is continuous by Theorem 3.5.1, and therefore the relations extend to $J \in \hat{\mathcal{B}}_k^r(U_1), K \in \hat{\mathcal{B}}_\ell^s(U_2)$, or $J \in \hat{\mathcal{B}}_k(U_1)$ and $K \in \hat{\mathcal{B}}_\ell(U_2)$.

(d): Suppose $J \hat{\times} K = 0$, $J \neq 0$ and $K \neq 0$. Let $\{e_i\}$ be an orthonormal basis of \mathbb{R}^{n+m} , respecting the Cartesian wedge product. Suppose e_I is a k -vector in $\Lambda_k(\mathbb{R}^n)$ and e_L is an ℓ -vector in $\Lambda_\ell(\mathbb{R}^m)$. Then

$$\int_Q \left(\int_P f de_I \right) g de_L = \int_{P \hat{\times} Q} h de_I de_L$$

where $h(x, y) = f(x)g(y)$, $P \in \mathcal{A}_k(U_1), Q \in \mathcal{A}_\ell(U_2)$. The proof follows easily by writing $P = \sum_{i=1}^r (p_i; \alpha_i)$ and $Q = \sum_{j=1}^s (q_j; \beta_j)$ and expanding. By continuity of Cartesian wedge product $\hat{\times}$ and the integral, we deduce

$$\int_K \left(\int_J f de_I \right) g de_L = \int_{J \hat{\times} K} h de_I de_L$$

where $h(x, y) = f(x)g(y)$, $J \in \hat{\mathcal{B}}_k(U_1), K \in \hat{\mathcal{B}}_\ell(U_2)$. Since $J \neq 0$ and $K \neq 0$, there exist $f \in \mathcal{B}_0^k(U_1), g \in \mathcal{B}_0(U_2)$ such that $\int_J f de_I \neq 0, \int_K g de_L \neq 0$. Therefore, $\int_K (\int_J f de_I) g de_L = \int_J f de_I \int_K g de_L \neq 0$, which implies $\int_{J \hat{\times} K} h de_I de_L \neq 0$, contradicting the assumption that $J \hat{\times} K = 0$.

(e): We first show that $(p; \sigma \otimes \alpha) \hat{\times} (q; \beta) = ((p, q); \sigma \otimes \alpha \wedge \beta)$. The proof is by induction on the order j of σ . This holds if $j = 0$, by definition of $\hat{\times}$. Assume it holds for order $j - 1$. Suppose that σ has order j . Let $\sigma = u \circ \sigma'$. By Proposition 9.1.3

$$\begin{aligned} (p; \sigma \otimes \alpha) \hat{\times} (q; \beta) &= (\lim_{t \rightarrow 0} (p + tu; \sigma' \circ \alpha/t) - (p; \sigma' \circ \alpha/t)) \hat{\times} (q; \beta) \\ &= \lim_{t \rightarrow 0} (p + tu; \sigma' \circ \alpha/t) \hat{\times} (q; \beta) - (p; \sigma' \circ \alpha/t) \hat{\times} (q; \beta) \\ &= \lim_{t \rightarrow 0} ((p + tu, q); \sigma' \circ \alpha \wedge \beta/t) - ((p, q); \sigma' \circ \alpha \wedge \beta/t) \\ &= P_u((p, q); \sigma' \circ \alpha \wedge \beta) = ((p, q); \sigma' \circ u \otimes \alpha \wedge \beta) = ((p, q); \sigma \circ \alpha \wedge \beta). \end{aligned}$$

In a similar way, one can show $(p; \sigma \otimes \alpha) \hat{\times} (q; \tau \otimes \beta) = ((p; q); \sigma \circ \tau \otimes \alpha \wedge \beta)$ using induction on the order j of σ .

(f): This follows from the definition of $\hat{\times}$ for simple elements:

$$\begin{aligned} (\pi^{1*}\omega \wedge \pi^{2*}\eta)((p; \alpha) \hat{\times} (q; \beta)) &= (\pi^{1*}\omega \wedge \pi^{2*}\eta)((p, q); \iota_{1*}\alpha \wedge \iota_{2*}\beta) \\ &= \omega(p; \alpha)\eta(q; \beta). \end{aligned}$$

It extends by linearity to Dirac chains, and by continuity to chains. \square

Cartesian wedge product extends to a continuous bilinear map $\hat{\times} : \hat{\mathcal{B}}_k(U_1) \times \hat{\mathcal{B}}_\ell(U_2) \rightarrow \hat{\mathcal{B}}_{k+\ell}(U_1 \times U_2)$ and the relations (b)-(f) continue to hold.

Corollary 9.1.6. *If $\{e_1, \dots, e_n\}$ is a basis of \mathbb{R}^n and $\{f_1, \dots, f_m\}$ is a basis of \mathbb{R}^m , then $\partial_{e_i}(J \hat{\times} K) = (\partial_{e_i} J) \hat{\times} K$ and $\partial_{f_j}(J \hat{\times} K) = (-1)^{\dim K} J \hat{\times} \partial_{f_j} K$ for all $J \in \hat{\mathcal{B}}_k^r(U_1)$, $K \in \hat{\mathcal{B}}_\ell^s(U_2)$, or $J \in \hat{\mathcal{B}}_k(U_1)$ and $K \in \hat{\mathcal{B}}_\ell(U_2)$.*

Proof. This follows from Theorem 9.1.5 since $\partial_{e_i} K = 0$ and $\partial_{f_j} J = 0$. \square

Example 9.1.7. *Recall that if A is affine k -cell in U_1 , then A is represented by an element $\tilde{A} \in \hat{\mathcal{B}}_k(U_1)$. If A and B are affine k - and ℓ -cells in \mathbb{R}^n and \mathbb{R}^m , respectively, then the classical Cartesian product $A \times B$ is an affine $(k+\ell)$ -cell in \mathbb{R}^{n+m} . The chain $\widetilde{A \times B}$ representing $A \times B$ satisfies $\widetilde{A \times B} = \tilde{A} \hat{\times} \tilde{B}$.*

Remarks 9.1.8.

- The boundary relations hold if we replace boundary ∂ with the directional boundary $\partial_v = P_v E_v^\dagger$ for $v \in \mathbb{R}^{n+m}$ where $v = (v_1, 0)$, $v_1 \in \mathbb{R}^n$, $0 \in \mathbb{R}^m$ or $v = (0, v_2)$, $0 \in \mathbb{R}^n$, $v_2 \in \mathbb{R}^m$.
- Cartesian wedge product is used to define a continuous convolution product on differential chains in a sequel [Har12a];
- According to Fleming ([Fle66], §6), “It is not possible to give a satisfactory definition of the cartesian product $A \times B$ of two arbitrary flat chains.” Fleming defines “Cartesian product” on polyhedral chains, but this coincides with our Cartesian wedge product according to Example 9.1.7. Cartesian wedge product is well-defined on Schwartz currents \mathcal{D}' (see [Fed69], §1.4.8).

10 Fundamental theorems of calculus for chains in a flow

10.1 Evolving chains

Let $J \in \hat{\mathcal{B}}_k^r(U)$ have compact support in U and $V \in \mathcal{V}^r(U)$ where U is open in \mathbb{R}^n . Let V_t be the time t map of the flow of V and $J_t := V_{t*} J$. For each $p \in U$, the image $V_t(p)$ is well-defined in U for sufficiently small t . Since J has compact support, there exists $t_0 > 0$ such that pushforward V_{t*} is defined on J for all $0 \leq t < t_0$. Let $\theta : U \times [0, t_0) \rightarrow U$ be given by $\theta(p, t) := V_t(p)$. Then $\theta \in \mathcal{M}^{r+1}(U \times [0, t_0), U)$. By Theorem 2.9.4, Corollary 6.5.8, Theorem 8.3.3, and Theorem 9.1.5 we infer

$$\{J_t\}_a^b := \theta_* E_{e_{n+1}}^\dagger (J \hat{\times} \widetilde{(a, b)}), \quad (10.1)$$

for $[a, b] \subset [0, t_0]$, is a well-defined element of $\hat{\mathcal{B}}_k^r(U \times (a, b))$ where $\widetilde{(a, b)}$ is the chain representing (a, b) , and $e_{n+1} \in \mathbb{R}^{n+1}$ is unit. We know of no way classical method to represent the entire collection of pushforward images of J as does $\{J_t\}_a^b$, a “moving picture” of J in the flow of the vector field V which maintains algebraic/geometric properties of J such as orientation, density, and dimension.

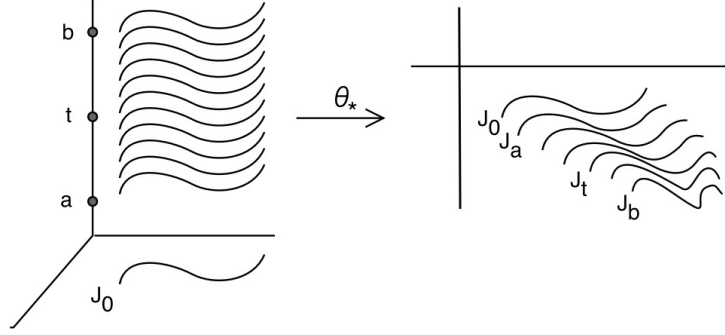


Figure 14: A chain in a flow

Examples 10.1.1. (a) Let $J = (p; 1)$ where $p \in U$ and X a Lipschitz vector field. Then $\{J_t\}_0^{t_0}$ represents the parametrized path of p along the integral curve of X through p and is an element of $\hat{\mathcal{B}}_0^1(U)$ while $\{(p; X(p))_t\}_0^{t_0}$ represents the one-dimensional version of the curve and is an element of $\hat{\mathcal{B}}_1^1(U)$.

(b) Let $A \in \mathcal{A}_0^s(U)$. For $s = 0$, then $\{A_t\}_0^{t_0}$ is the sum of representatives of finitely many parametrized paths starting at the points in the support of A . Of course, t_0 must be chosen so that each path is contained in U . For $s = 1$, we obtain the path of the support of a dipole $(p; u \otimes 1)$ as it moves in U .

(c) For $A' \in \mathcal{A}_n^s(U)$, then pushforward modifies its n -dimensional mass $(F_*(p; \alpha) = (F(p); F_{p*}\alpha)$. However, $\perp F_* \perp (p; \alpha) = (F(p); \alpha)$ for all n -vectors α . The latter is useful for modeling incompressible fluids. Of course, one can also consider $A \in \mathcal{A}_k^s(U)$ for any $0 \leq k \leq n$ and $s \geq 0$ and model particles, including dipoles of any order. Fluids which are partially compressible can be modeled by using a linear combination of the two approaches $c_1 \perp F_* \perp + c_2 F_*$. For $0 < k < n$, then $\perp F_* \perp$ modifies a simple k -element $(p; \alpha)$, i.e., changes its mass and k -direction, according to the infinitesimal change of mass and direction of the simple $(n - k)$ -element normal to $(p; \alpha)$ while F_* alters mass and direction according to how F_* acts on $(p; \alpha)$ itself.

We can take linear combinations to model a flowing material which is partly compressible: $c\theta_*E_{e_{n+1}}^\dagger(J \hat{\times} \widetilde{(a, b)}) + (1 - c) \perp \theta_* \perp_t E_{e_{n+1}}^\dagger(J \hat{\times} \widetilde{(a, b)})$ where \perp_t is \perp restricted to the slice of \mathbb{R}^{n+1} at time t .

(d) Let \widetilde{M} represent a submanifold M of U . Then $\{\widetilde{M}_t\}_0^{t_0}$ is the differential chain representing the evolution of M , and takes into account infinitesimal distortions induced by the flow of X .

The construction of an evolving chain and the results below extend to $J \in \hat{\mathcal{B}}_k(U)$ with compact support in U , and $V \in \mathcal{V}(U)$ of class \mathcal{B}^∞ . It is not difficult to define evolving chains in space-time and the author is developing extensions of Reynolds' transport and its applications from this viewpoint in [Har12c].

Lemma 10.1.2. The following relations hold:

$$(a) \theta_*(J \hat{\times}(t; 1)) = V_{t*}J;$$

$$(b) \quad P_V \theta_* = \theta_* P_{e_{n+1}};$$

$$(c) \quad \partial \{J_t\}_a^b = \{\partial J_t\}_a^b;$$

Proof. (a): It suffices to prove this for $J = (p; \alpha)$, a simple k -element with $p \in U$. But $\theta_*((p; \alpha) \hat{\times} (t; 1)) = \theta_*((p, t); \iota_{1*} \alpha) = V_{t*}(p; \alpha)$ since $\theta(p, t) = V_t(p)$.

(b): By Theorem 8.5.1 (c), $\theta_* P_{e_{n+1}} = P_{\theta e_{n+1}} \theta_* = P_V \theta_*$

(c):

$$\begin{aligned} \partial \{J_t\}_a^b &= \partial \theta_* E_{e_{n+1}}^\dagger (J \hat{\times} \widetilde{(a, b)}) && \text{by (10.1)} \\ &= \theta_* \partial E_{e_{n+1}}^\dagger (J \hat{\times} \widetilde{(a, b)}) && \text{by Proposition 6.7.4} \\ &= \theta_* E_{e_{n+1}}^\dagger \partial (J \hat{\times} \widetilde{(a, b)}) && \text{by (c)} \\ &= \theta_* E_{e_{n+1}}^\dagger (\partial J \hat{\times} \widetilde{(a, b)} - J \hat{\times} ((b; 1) - (a; 1))) && \text{by Theorem 9.1.5(b)} \\ &= \{(\partial J)_t\}_a^b && \text{by (10.1) applied to } \partial J \text{ and} \\ &&& \text{since } E_{e_{n+1}}^\dagger (J \hat{\times} ((b; 1) - (a; 1))) = 0. \end{aligned}$$

□

10.2 Fundamental theorems

Given $(a, b) \subset \mathbb{R}$, $t \in \mathbb{R}$, and a differential chain J , let J_a, J_b and J_t be as in §10.1.

Theorem 10.2.1 (Fundamental theorems for chains in a flow). *Suppose $J \in \hat{\mathcal{B}}_k^r(U)$ is a differential chain with compact support in U , $V \in \mathcal{V}^r(U)$ is a vector field, and $\omega \in \mathcal{B}_k^{r+1}(U)$ is a differential form where U is open in \mathbb{R}^n . Then*

$$\int_{J_b} \omega - \int_{J_a} \omega = \int_{\{J_t\}_a^b} L_V \omega = \int_a^b \left(\int_{J_t} L_V \omega \right) dt.$$

Proof. Since Dirac chains are dense, and both ∂_v and Cartesian wedge product are continuous for all $v \in \mathbb{R}^n$, we use Corollary 9.1.6 to conclude

$$\partial_{e_1}(\widetilde{(a, b)} \hat{\times} J) = (b; 1) \hat{\times} J - (a; 1) \hat{\times} J.$$

Finally we change coordinates to consider a local flow chart of V : By Proposition 8.5.1 we know $\partial_{\theta e_1} \theta_* = \theta_* \partial_{e_1}$.

Using Corollary 3.2.5 we deduce

$$\begin{aligned}
\int_{\{J_t\}_a^b} L_V \omega &= \int_{P_V \{J_t\}_a^b} \omega && \text{by Theorem 8.4.2} \\
&= \int_{P_V \theta_* E_{e_{n+1}}^\dagger (J \hat{\times} (\widetilde{a, b}))} \omega && \text{by Definition 10.1} \\
&= \int_{\theta_* P_{e_{n+1}} E_{e_{n+1}}^\dagger (J \hat{\times} (\widetilde{a, b}))} \omega && \text{by Proposition 8.5.1} \\
&= \int_{\theta_* \partial_{e_{n+1}} (J \hat{\times} (\widetilde{a, b}))} \omega && \text{by the definition of directional boundary p.27} \\
&= \int_{\theta_* (J \hat{\times} (b; 1)) - \theta_* (J \hat{\times} (a; 1))} \omega && \text{since } \partial(\widetilde{a, b}) = (b; 1) - (a; 1) \\
&= \int_{V_{b*} J} \omega - \int_{V_{a*} J} \omega && .
\end{aligned}$$

We next establish the second equality:

Let $f(t) = \int_{J_t} \omega$. It suffices to show that $f'(t) = \int_{P_V J_t} \omega$. The result will then follow from the fundamental theorem of calculus. Since P_V is continuous, $P_V J_t = \lim_{h \rightarrow 0} \frac{J_{t+h} - J_t}{h}$ for each $t \in [a, b]$. Hence

$$f'(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \rightarrow 0} \int_{(J_{t+h} - J_t)/h} \omega = \int_{P_V J_t} \omega$$

as we hoped. Then $\int_a^b \left(\int_{P_V J_t} \omega \right) dt = \int_{J_b} \omega - \int_{J_a} \omega$. The result follows by Theorem 8.4.3. \square

If J_a is a 0-element, then this is the fundamental theorem of calculus for integral curves.

Theorem 10.2.2 (Stokes' theorem for evolving chains). *Let $\omega \in \mathcal{B}_{k-1}^{r+1}(U)$ be a differential form, $J \in \hat{\mathcal{B}}_k^r(U)$ a differential chain, and $V \in \mathcal{V}^r(U)$ a vector field. Then*

$$\int_{\{J_t\}_a^b} dL_V \omega = \int_{\partial J_b} \omega - \int_{\partial J_a} \omega.$$

Proof. This follows directly from Stokes' Theorem 3.5.4, Theorem 10.2.1, and Lemma 10.1.2 (c). \square

Corollary 10.2.3. *Let $\omega \in \mathcal{B}_{k-1}^{r+1}(U)$ be a differential form, $J \in \hat{\mathcal{B}}_k^r(U)$ a differential chain, and $V \in \mathcal{V}^r(U)$ a vector field. If $L_V \omega$ is closed, then $\int_{\partial J_b} \omega = \int_{\partial J_a} \omega$.*

10.3 Differentiation of the integral

Definition 10.3.1. *We say that a one-parameter family of differential k -chains $J_t \in \hat{\mathcal{B}}_k^r(U), r \geq 0$, or $J_t \in \hat{\mathcal{B}}_k(U)$, is **differentiable at time t** if $\frac{\partial}{\partial t} J_t = \lim_{h \rightarrow 0} (J_{t+h} - J_t)/h$ is well-defined in $\hat{\mathcal{B}}_k^r(U)$, respectively $\hat{\mathcal{B}}_k(U)$. Similarly, a one-parameter family of differential k -forms $\omega_t \in \mathcal{B}_k^r(U), r \geq 0$, or $\omega_t \in \mathcal{B}_k(U)$ is differentiable at time t if $\frac{\partial}{\partial t} \omega_t = \lim_{h \rightarrow 0} (\omega_{t+h} - \omega_t)/h$ is well-defined in $\mathcal{B}_k^r(U)$, respectively $\mathcal{B}_k(U)$.*

Example 10.3.2. Suppose $f_1 : [0, 1] \rightarrow \mathbb{R}$ is a Lebesgue integrable function and $f_t : [0, 1] \rightarrow \mathbb{R}$ is a monotone increasing sequence of step functions converging to f . Let J_t be the differential 1-chain representing the graph of f_t , $0 \leq t < 1$. For convenience, set $f_t = f_1$ for $1 \leq t \leq 2$. It follows that any subsequence J_{t_j} with increasing $t_j \leq 1$ is Cauchy in $\hat{\mathcal{B}}_1^1(\mathbb{R}^2)$. Let $J_1 = \lim_{j \rightarrow \infty} J_{t_j}$. While $\text{supp}(J_1)$ is the graph of f_1 , $\text{supp}(\partial J_1)$ is the set of discontinuity points in the graph of f_1 , together with the endpoints. Furthermore $\frac{\partial}{\partial t} J_s$ exists in $\hat{\mathcal{B}}_1^2(\mathbb{R}^2)$ and is a 1-chainlet of dipole order 1 for each $0 < s < 2$.

Numerous other examples are provided using flows. If J_t is an *evolving differential k -chain*, that is, the pushforward of a k -chain J via the time t map of a smooth vector field V , flow, it follows directly from the definitions that J_t is differentiable at time t for each t of definition, and

$$\frac{\partial}{\partial t} J_t = P_V J_t. \quad (10.2)$$

Our next result extends the classical Leibniz Integral Rule to families of differential chains which are differentiable in time.

Theorem 10.3.3 (Generalized Leibniz Integral Rule). *If $J_t \in \hat{\mathcal{B}}_k^r(U)$ and $\omega_t \in \mathcal{B}_k^r(U)$ are differentiable in time, then*

$$\frac{\partial}{\partial t} \int_{J_t} \omega_t = \int_{J_t} \frac{\partial}{\partial t} \omega_t + \int_{\frac{\partial}{\partial t} J_t} \omega_t.$$

Proof.

$$\frac{\partial}{\partial t} \int_{J_t} \omega_t = \lim_{h \rightarrow 0} \int_{\frac{J_{t+h} - J_t}{h}} \omega_t + \int_{J_t} \frac{\omega_{t+h} - \omega_t}{h} = \int_{\frac{\partial}{\partial t} J_t} \omega_t + \int_{J_t} \frac{\partial}{\partial t} \omega_t.$$

□

If J_t is constant with respect to time, we get $\frac{\partial}{\partial t} \int_{J_t} \omega_t = \int_{J_t} \frac{\partial}{\partial t} \omega_t$, as expected. This result holds for all families of differential chains and differential forms that are differentiable in time. For chains evolving under a flow, we obtain powerful generalizations of Differentiation of the Integral and the Reynolds' Transport theorem of continuum mechanics since J_0 can be any differential chain and V any sufficiently smooth vector field.

For evolving chains, we immediately deduce from Equation 10.2, Corollary 10.3.3 and the duality $L_V \omega = \omega P_V$ (see §3.3):

Theorem 10.3.4 (Differentiating the Integral). *Suppose $J_t \in \mathcal{B}_k^r(U)$ is an evolving differential k -chain under the flow of a vector field $V \in \mathcal{V}^{r+1}(U)$, and $\omega_t \in \mathcal{B}_k^{r+1}(U)$ is a smooth differential k -form differentiable in time. Then*

$$\frac{\partial}{\partial t} \int_{J_t} \omega_t = \int_{J_t} \left(\frac{\partial}{\partial t} \omega_t + L_V \omega_t \right).$$

The classical method of differentiating the integral has been extremely useful in engineering and physics²⁵. From the

²⁵Feynman wrote in his autobiography [Fey85], “That book [Advanced Calculus, by Wood.] also showed how to differentiate parameters under the integral sign - it’s a certain operation. It turns out that’s not taught very much in the universities; they don’t emphasize it. But I caught on how to use that method, and I used that one damn tool again and again. So because I was self-taught using that book, I had peculiar methods of doing integrals. The result was, when guys at MIT or Princeton had trouble doing a certain integral, it was because they couldn’t do it with the standard methods they had learned in school. If it was contour integration, they would have found it; if it was a simple series expansion, they would have found it. Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else’s, and they had tried all their tools on it before giving the problem to me.”

duality $d\omega = \omega\partial$ (see Theorem 3.5.4) and Cartan's Magic Formula $L_V = di_V + i_Vd$ we immediately deduce:

Corollary 10.3.5. *If ω_t is closed, then*

$$\frac{\partial}{\partial t} \int_{J_t} \omega_t = \int_{J_t} \frac{\partial}{\partial t} \omega_t + \int_{\partial J_t} i_V \omega_t.$$

If J_t is a cycle, then

$$\frac{\partial}{\partial t} \int_{J_t} \omega_t = \int_{J_t} \frac{\partial}{\partial t} \omega_t + \int_{J_t} i_V d\omega_t.$$

Corollary 10.3.6 (Generalization of the Reynolds' Transport Theorem to nonsmooth domains, evolving in a flow). *Suppose $J_t \in \mathcal{B}_k^r(U)$ is an evolving differential chain under the flow of a vector field $V \in \mathcal{V}^{r+1}(U)$, and $\omega_t \in \mathcal{B}_k^{r+1}(U)$ is a smooth differential form which is differentiable in time. Then*

$$\frac{\partial}{\partial t} \left[\int_{J_t} \omega_t \right] = \int_{J_t} \frac{\partial}{\partial t} \omega_t + \int_{\partial J_t} i_X \omega_t + \int_{E_X J_t} d\omega_t = \int_{J_t} \frac{\partial}{\partial t} \omega_t + \int_{P_X J_t} \omega_t.$$

The classical version, a cornerstone of mechanics and fluid dynamics, is a straightforward consequence. Our result applies to all domains of integration represented by differential chains.

11 Differential chains in manifolds

The Levi-Civita connection and metric can be used to extend this theory to open subsets of Riemannian manifolds (see [Pug09], as well). We sketch some of the main ideas here. Let $\mathcal{R}_{U,M}$ be the category whose objects are pairs (U, M) of open subsets U of Riemannian manifolds M , and morphisms are smooth maps. Let TVS be the category whose objects are locally convex topological vector spaces, and morphisms are continuous linear maps between them. Then $\hat{\mathcal{B}}$ is a functor from $\mathcal{R}_{U,M}$ to TVS . We define $\hat{\mathcal{B}}(U) = \hat{\mathcal{B}}(U, M)$ much as we did for $M = \mathbb{R}^n$, but with a few changes: Norms of tangent vectors and masses of k -vectors are defined using the metric. The connection can be used to define B^r norms of vector fields. Dirac k -chains are defined as formal sums $\sum (p_i; \alpha_i)$ where $p_i \in U$, $\alpha_i \in \Lambda_k(T_{(p_i)}(M))$. The vector space of Dirac k -chains is $\mathcal{A}_k(U, M)$. Difference chains are defined using pushforward along the flow of locally defined, unit vector fields. The symmetric algebra is replaced by the universal enveloping algebra. From here, we can define the B^r norms on $\mathcal{A}_k(U, M)$, and the B^r norms on forms, and thus the topological vector spaces $\hat{\mathcal{B}}_k(U, M)$ and $\mathcal{B}_k(U, M)$. Support of a chain is a subset of M together with the attainable boundary points of M . Useful tools include naturality of the operators §8.5, partitions of unity §4.2, and cosheaves.

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